About dual two-dimensional oscillator and Coulomb-like theories on pseudosphere

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Abstract

We present a mathematically rigorous quantum-mechanical treatment of a two-dimensional nonrelativistic quantum dual theories (with oscillator and Coulomb like potentials) on pseudosphere and compare their spectra and the sets of eigenfunctions. We construct all self-adjoint Schrodinger operators for these theories and represent rigorous solutions of the corresponding spectral problems. Solving the first part of the problem, we use a method of specifying s.a. extensions by (asymptotic) s.a. boundary conditions. Solving spectral problems, we follow the Krein's method of guiding functionals. We show, that there is one to one correspondence between the spectral points of dual theories in the planes energy-coupling constants not only for discrete, but also for continuous spectra.

1 Introduction

It is well known [1], that if one introduces in a radial part of the D dimensional oscillator $(D>2)$

$$\frac{d^2R}{du^2}+\frac{D-1}{u}\frac{dR}{du}-\frac{L\,(L+D-2)}{u^2}R+\frac{2\mu}{\hbar^2}\left(E-\frac{\mu\omega^2u^2}{2}\right)R=0 \tag{1.1}$$

(here R is the radial part of the wave function for the D dimensional oscillator $(D>2)$ and $L=0,1,2,...$ are the eigenvalues of the global angular momentum) $r=u^2$ then equation (1.1) transforms into

$$\frac{d^2R}{dr^2}+\frac{d-1}{r}\frac{dR}{dr}-\frac{l\,(l+d-2)}{r^2}R+\frac{2\mu}{\hbar^2}\left(\mathcal{E}+\frac{\alpha}{r}\right)R=0 \tag{1.2}$$

where $d=D/2+1 \quad l=L/2 \quad \mathcal{E}=-\frac{\mu\omega^2}{8} \quad \alpha=E/4$, which formally is identical to the radial equation for d-dimensional hydrogen atom.

Equations (1.1) and (1.2) are dual to each other and the duality transformation is $r=u^2$. For discreet spectrum of these equations (and wave fuctions regular at the origin) it was proved, that to each state of equation (1.1) corresponds a state in (1.2), and visa versa [2, 3]. However the correspondence of the states in general (for discrete, as well as continuous spectra and for all values of the parameters of the theory) the problems was not considered.

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In [4] we constructed all self-adjoint Schrodinger operators for nonrelativistic one-dimensional quantum dual theories and represented rigorous solutions of the corresponding spectral problems. We have shown that there is one to one correspondence between the spectra of dual theories for discreet , as well as continuous spectra.

In this paper will solve the quantum problem of two dimensional quantum dual theories (with oscillator and Coulomb like potentials) on pseudosphere and compare their spectra and the sets of eigenfunctions. As it was in one dimensional case, we again have a correspondence of the states for all values of the parameters E_O, λ, E_C, and g, except when the angular momentum $m = 1$, when the duality is one-to-one only in the case of parameter of s.a. extension $\zeta = \pi/2$ (see below in coulomb case). The interest to these models was stimulated also by the fact that among the theorists dealing with similar problems exists a notion, that the "Hamiltonian isn't self adjoit at high energies" [5]. In section 2 we will consider the quantum problem for the oscillator, will find solutions of the equation for all values of the variable and parameters. In Section 3 we will consider the quantum problem for Coulomb-like system. The results will be compared in section 4, where we will show the one-to one correspondense of the spectra and proper functions of the Hamiltonians of both problems.

2 Quantum two-dimentional oscillator-like interaction on pseudosphere

2.1 Preliminaries

Here, we consider the QM of a paticle moving on pseudosphere (two-sheet hyperboloid) in an "oscillator" potential. We describe the pseudosphere by the coordinates $\mathbf{u} = \{u^1, u^2\}$ of its stereographic projection on the plane, such that the coordinate $\mathbf{s} = \{s^1, s^2, s^3\}$ of the ambient space, $(s^3)^2 - (s^1)^2 - (s^2)^2 = R^2$ are

$$s^1 = \frac{2R^2u^1}{R^2 - r^2},\ s^2 = \frac{2R^2u^2}{R^2 - r^2},\ s^3 = R\frac{R^2 + r^2}{R^2 - r^2},\ r = \sqrt{(u^1)^2 + (u^2)^2},$$

where R is a radius of the pseudosphere.The disc $r < R$ describes the upper sheet $s^3 > R$, the exterior of the disc $r > R$ describes the lower sheet $s^3 < -R$. There is an useful one-to-one map P, $P^2 = 1$, of the interior of disc onto the exterior of disc and inversely:

$$P\mathbf{u} = \mathbf{u}_P = \{u_P^1, u_P^2\} = \left\{\frac{R^2}{r^2}u^1, \frac{R^2}{r^2}u^2\right\},\ r_P = \frac{R^2}{r},\ P^2\mathbf{u} = \mathbf{u},$$

We will call this map by the parity transformation and the operator P by the parity operator. Note that P-transformation commutes with the rotation of u-plane around origin.

The metric $g_{jk}(\mathbf{u})$, the invariant volume element $d\Lambda(\mathbf{u})$, and the Beltrami-Laplace operator $\Delta_{BL}(\mathbf{u})$ have the following form in the coordinates $\mathbf{u}$:

$$g_{jk}(\mathbf{u}) = \frac{R^4}{(R^2 - r^2)^2}\delta_{jk},\ g^{jk}(\mathbf{u}) = \frac{(R^2 - r^2)^2}{R^4}\delta^{jk},\ \sqrt{g(\mathbf{u})} = \frac{R^4}{(R^2 - r^2)^2},$$
$$d\Lambda(\mathbf{u}) = \frac{R^4}{(R^2 - r^2)^2}d^2u,\ \Delta_{BL}(\mathbf{u}) = \frac{(R^2 - r^2)^2}{R^4}\Delta(\mathbf{u}),\ \Delta(\mathbf{u}) = \partial_{u^k}\partial_{u^k},\ j, k = 1, 2.$$

The wave functions $\Psi(\mathbf{u})$ of the QM-problem under consideration belong to the Hilbert space $\mathfrak{H} = L^2_\Lambda(\mathbb{R}^2)$ with the scalar product

$$(\Psi_1, \Psi_2) = \int_{\mathbb{R}^2} \overline{\Psi_1(\mathbf{u})}\Psi_2(\mathbf{u})d\Lambda(\mathbf{u}),\ \forall\Psi_1, \Psi_2 \in \mathfrak{H},$$

and the s.a. Hamiltonians are associated with a differential operation $\check{H}$,

$$\check{H} = \check{H}(\mathbf{u}) = -\Delta_{BL}(\mathbf{u}) + V(\mathbf{u}),\ V(\mathbf{u}) = \frac{4(q-1)r^2}{(R^2+r^2)^2},\ q = \frac{R^4}{4}\lambda,$$

where $V(\mathbf{u})$ is an "oscillator" potential and q is a coupling constant (λ is a coupling constant in the plane limit $R \to \infty$) . We note that the introduced invariant volume element, the Beltrami-Laplace operator, "oscillator" potential, differential operation and scalar product are invariant under parity transformation:

$$\begin{aligned}
&Pd\Lambda(\mathbf{u}) = d\Lambda(\mathbf{u}_P) = d\Lambda(\mathbf{u}),\ P\Delta_{BL}(\mathbf{u})P = \Delta_{BL}(\mathbf{u}_P) = \Delta_{BL}(\mathbf{u}),\\
&PV(\mathbf{u}) = V(\mathbf{u}_P) = V(\mathbf{u}),\ P\check{H}(\mathbf{u})P = \check{H}(\mathbf{u}_P) = \check{H}(\mathbf{u}),\ (\Psi_{1P}, \Psi_{2P}) = (\Psi_1, \Psi_2)\,,
\end{aligned}$$

where $\Psi_P(\mathbf{u}) = P\Psi(\mathbf{u}) = \Psi(\mathbf{u}_P)$. Furthermore, a relation holds:

$$\int_{r_<<R} \overline{\Psi_1(\mathbf{u}_<)}\Psi_2(\mathbf{u}_<)d\Lambda(\mathbf{u}_<) = \int_{r_>>R} \overline{\tilde{\Psi}_1(\mathbf{u}_>)}\tilde{\Psi}_2(\mathbf{u}_>)d\Lambda(\mathbf{u}_>),$$

where $\mathbf{u}_< = P\mathbf{u}_>$, $\tilde{\Psi}_k(\mathbf{u}_<) = \Psi_{kP}(\mathbf{u}_>) = \Psi_k(P\mathbf{u}_>)$.

2.1.1 Polar coordinates r, φ

Rewrite the above introduced quantities in the polar coordinates r, φ, $u^1 = r\cos\varphi$, $u^2 = r\sin\varphi$. We have:

$$\begin{aligned}
&d\Lambda(\mathbf{u}) = d\omega(r)d\varphi,\ d\omega(r) = \frac{R^4 r}{(R^2-r^2)^2}dr,\\
&\check{H} = -\Delta_{BLr} - \Delta_{BL\varphi} + V(r),\ \Delta_{BL\varphi} = \frac{(R^2-r^2)^2}{R^4r^2}\partial_\varphi^2,\ V(r) = \frac{4(q-1)r^2}{(R^2+r^2)^2},\\
&\Delta_{BLr} = \frac{(R^2-r^2)^2}{R^4}\Delta_r,\ \Delta_r = \frac{1}{r}\partial_r r\partial_r,
\end{aligned}$$

$$\begin{aligned}
&\Psi(\mathbf{u}) = \sum_{m\in Z}\Lambda_m(\mathbf{u}),\ \Lambda_m(\mathbf{u}) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}\Psi_m(r),\\
&(\Lambda_{1m_1}, \Lambda_{2m_2}) = \delta_{m_1m_2}\left(\Lambda_{1m_1}\Lambda_{2m_1}\right),\ (\Lambda_{1m}, \Lambda_{2m}) = \int_0^\infty \overline{\Psi_{1m}(r)}\Psi_{2m}(r)d\omega(r)\\
&\check{H}\Psi(\mathbf{u}) = \sum_{m\in Z}\frac{1}{\sqrt{2\pi}}e^{im\varphi}\check{H}_m\Psi_m(r),\ \check{H}_m = -\Delta_{BLr} + \frac{m^2(R^2-r^2)^2}{R^4r^2} + V(r).
\end{aligned}$$

Represent $\Psi_m(r)$ in the form

$$\Psi_m(r) = \frac{|R^2-r^2|}{R^2\sqrt{r}}\psi_m(r).$$

Then we have

$$\begin{aligned}
&(\Lambda_{1m},\Lambda_{2m})=\langle\psi_{1m},\psi_{2m}\rangle=\int_0^\infty\overline{\psi_{1m}(r)}\psi_{2m}(r)dr,\\
&\check{H}_m\Psi_m(r)=\frac{|R^2-r^2|}{R^2\sqrt{r}}\check{h}_m\psi_m(r),\\
&\check{h}_m=\frac{\sqrt{r}}{R^2-r^2}\check{H}_m\frac{R^2-r^2}{\sqrt{r}}=\\
&=-\partial_r p_0(r)\partial_r+\frac{10R^2r^2-9r^4-R^4+4m^2(R^2-r^2)^2}{4R^4r^2}+V(r),\ p_0(r)=\frac{(R^2-r^2)^2}{R^4}.
\end{aligned}\tag{2.1}$$

2.1.2 P-transformation

In the polar coordinates, we have

$$Pr=r_P=\frac{R^2}{r},\ P\varphi=\varphi_P=\varphi.$$

$$\begin{aligned}
&Pr\partial_r=r_P\partial_{r_P}=-\frac{R^2}{r}\frac{r^2\partial_r}{R^2}=-r\partial_r,\ P\frac{dr}{r}=\frac{dr_P}{r_P}=-\frac{r}{R^2}\frac{R^2dr}{r^2}=-\frac{dr}{r},\\
&I_\pm(r)=\frac{(R^2\pm r^2)^2}{R^4r^2},\ PI_\pm(r)=I_\pm(r_P)=I_\pm(r),\\
&d\omega(r)=\frac{1}{I_-(r)}\frac{dr}{r},\ Pd\omega(r)=d\omega(r),\ d\Lambda(\mathbf{u})=\frac{1}{I_-(r)}\frac{dr}{r}d\varphi,\ Pd\Lambda(\mathbf{u})=d\Lambda(\mathbf{u}),\\
&\Delta_{BLr}=\frac{1}{4}I_-(r)(r\partial_r)^2,\ P\Delta_{BLr}(r)=\Delta_{BLr}(r_P)=\Delta_{BLr}(r),\\
&\Delta_{BL\varphi}=I_-(r)\partial^2_\varphi,\ P\Delta_{BL\varphi}(\mathbf{u})=\Delta_{BL\varphi}(\mathbf{u}_P)=\Delta_{BL\varphi}(\mathbf{u}),\\
&V(r)=\frac{4(q-1)}{R^4I_+(r)},\ PV(r)=V(r_P)=V(r),\\
&P\check{H}(\mathbf{u})P=\check{H}(\mathbf{u}_P)=\check{H}(\mathbf{u}),\ P\check{H}_m(r)P=\check{H}_m(r_P)=\check{H}_m(r).
\end{aligned}$$

Operator P acts on the radial wave finctions by the following way (subscript “m” is omitted),

$$P\Psi(r)=\Psi(r_P(r))=\Psi(R^2/r).$$

It is convenient to introduce some notation:

$$\begin{aligned}
&r=\{\rho,\xi\},\ 0\le\xi\le R,\ R\le\rho<\infty,\\
&P_{\rho\xi}\xi=\rho_P=\frac{R^2}{\rho},\ P_{\xi\rho}\rho=\xi_P=\frac{R^2}{\xi},\ \Psi(r)=\begin{pmatrix}\Psi^>(\rho)\\ \Psi^<(\xi)\end{pmatrix},\\
&P_{\rho\xi}P_{\xi\rho}\rho=P_{\rho\xi}\xi_P=P_{\rho\xi}\frac{R^2}{\xi}=\frac{R^2}{\rho_P}=\rho,\ P_{\xi\rho}P_{\rho\xi}\xi=\xi,
\end{aligned}$$

$$\begin{aligned}
&P_{\rho\xi}\Psi^<(\xi)=\Psi^<(R^2/\rho),\ P_{\xi\rho}\Psi^>(\rho)=\Psi^>(R^2/\xi),\ \check{H}(r)=\begin{pmatrix}\check{H}^>(\rho)&0\\0&\check{H}^<(\xi)\end{pmatrix},\\
&P\Psi(r)=\Psi_P(r)=\begin{pmatrix}\Psi_P^{|>}(\rho)\\ \Psi_P^{|<}(\xi)\end{pmatrix}\Longrightarrow\Psi_P^{|>}(\rho)=\Psi^<(R^2/\rho)=\Psi^<(\rho_P),\\
&\Psi_P^{|<}(\xi)=\Psi^>(R^2/\xi)=\Psi^>(\xi_P)\Longrightarrow P=\begin{pmatrix}0&P_{\rho\xi}\\P_{\xi\rho}&0\end{pmatrix},
\end{aligned}$$

where $\Psi_P^{|>}(\rho) \equiv (\Psi_P)^>(\rho)$ and so forth.

Checking:

$$\check{H}(r) = P\check{H}(r)P = \begin{pmatrix} P_{\rho\xi}\check{H}^<(\xi)P_{\xi\rho} & 0 \\ 0 & P_{\xi\rho}\check{H}^>(\rho)P_{\rho\xi} \end{pmatrix} =$$
$$= \begin{pmatrix} \check{H}^<(\rho_P) & 0 \\ 0 & \check{H}^>(\xi_P) \end{pmatrix} = \check{H}(r_P) = \check{H}(r).$$

Turn out to the "h-space". We have

$$\Psi(r) = \frac{|R^2-r^2|}{R^2\sqrt{r}}\psi(r),\ P\Psi(r) = \frac{|R^2-r_P^2|}{R^2\sqrt{r_P}}\psi(r_P) = \frac{|R^2-r^2|}{Rr^{3/2}}\psi(r_P) =$$
$$= \frac{|R^2-r^2|}{R^2\sqrt{r}}\tilde{P}\psi(r) \Longrightarrow \tilde{P}\psi(r) = \frac{R}{r}\psi(r_P) = P\frac{r}{R}\psi(r) \Longrightarrow$$
$$\Longrightarrow \tilde{P} = \frac{R}{r}P = \frac{r_P}{R}P = P\frac{r}{R} = .\begin{pmatrix} 0 & \frac{R}{\rho}P_{\rho\xi} \\ \frac{R}{\xi}P_{\xi\rho} & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & \frac{\rho_P}{R}P_{\rho\xi} \\ \frac{\xi_P}{R}P_{\xi\rho} & 0 \end{pmatrix} = \begin{pmatrix} 0 & P_{\rho\xi}\frac{\xi}{R} \\ P_{\xi\rho}\frac{\rho}{R} & 0 \end{pmatrix}.$$

$$\tilde{P}^2\psi(r) = \begin{pmatrix} \frac{R}{\rho}\frac{R}{\rho_P}P_{\rho\xi}P_{\xi\rho} & 0 \\ 0 & \frac{R}{\xi}\frac{R}{\xi_P}P_{\xi\rho}P_{\rho\xi} \end{pmatrix}\begin{pmatrix} \psi^>(\rho) \\ \psi^<(\xi) \end{pmatrix} = \begin{pmatrix} \psi^>(\rho) \\ \psi^<(\xi) \end{pmatrix} = \psi(r),$$

$$\tilde{P}\check{h}(r)\tilde{P} = \frac{R}{r}\check{h}(r_P)\frac{r}{R} = \frac{R}{r}P\frac{\sqrt{r}}{R^2-r^2}\check{H}_m(r)\frac{R^2-r^2}{\sqrt{r}}P\frac{r}{R} =$$
$$= \frac{R}{r}\frac{r^{3/2}}{R(R^2-r^2)}\check{H}_m(r)\frac{R(R^2-r^2)}{r^{3/2}}\frac{r}{R} = \frac{\sqrt{r}}{R^2-r^2}\check{H}_m(r)\frac{R^2-r^2}{\sqrt{r}} = \check{h}(r), \tag{2.2}$$

$$\psi(r) = \begin{pmatrix} \psi^>(\rho) \\ \psi^<(\xi) \end{pmatrix},\ \tilde{P}\psi(r) = P\frac{r}{R}\psi(r) = \psi_{\tilde{P}}(r) = \begin{pmatrix} \psi_{\tilde{P}}^{|>}(\rho) \\ \psi_{\tilde{P}}^{|<}(\xi) \end{pmatrix} \Longrightarrow$$
$$\Longrightarrow \psi_{\tilde{P}}^{|>}(\rho) = P_{\rho\xi}\frac{\xi}{R}\psi^<(\xi) = \frac{\rho_P}{R}\psi^<(\rho_P),$$
$$\psi_{\tilde{P}}^{|<}(\xi) = P_{\xi\rho}\frac{\rho}{R}\psi^>(\rho) = \frac{\xi_P}{R}\psi^>(\xi_P).$$

$$\check{h}(r) = .\begin{pmatrix} \check{h}^>(\rho) & 0 \\ 0 & \check{h}^<(\xi) \end{pmatrix},\ \check{h}^>(\rho) = \frac{R}{\rho}\check{h}^<(\rho_P)\frac{\rho}{R},\ \check{h}^<(\xi) = \frac{R}{\xi}\check{h}^>(\xi_P)\frac{\xi}{R}, \tag{2.3}$$

two last equalities follow from eq. (2.2),

$$\check{h}^>(\rho)\psi_{\tilde{P}}^{|>}(\rho) = \frac{R}{\rho}\check{h}^<(\rho_P)\frac{\rho}{R}\frac{\rho_P}{R}\psi^<(\rho_P) = \frac{R}{\rho}\check{h}^<(\rho_P)\psi^<(\rho_P) = \frac{R}{\rho}P_{\rho\xi}\check{h}^<(\xi)\psi^<(\xi), \tag{2.4}$$

$$[\check{h}^<(\xi) - W]\psi^<(\xi) = 0 \Longrightarrow [\check{h}^>(\rho) - W]\psi_{\tilde{P}}^{|>}(\rho) = 0 \tag{2.5}$$

2.1.3 Some relations

Any functions $\Lambda_m(\mathbf{u})=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\Psi_m(r)$ can be represented in the form

$$\begin{aligned}
&\Lambda_m(\mathbf{u})=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\Psi_{m,+}(r)+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\Psi_{m.-}(r),\\
&\Psi_{m,\zeta}(r)=\frac{1+\zeta P}{2}\Psi_m(r),\ P\Psi_{m,\zeta}(r)=\zeta\Psi_{m,\zeta}(r),\ \zeta=\pm,\\
&\Psi_{m,\zeta}(r)=\left(\begin{array}{c}\Psi^{>}_{m,\zeta}(\rho)\\ \Psi^{<}_{m,\zeta}(\xi)\end{array}\right),\ \Psi^{<}_{m,\zeta}(\xi)=\frac{1}{2}\left[\Psi^{<}_m(\xi)+\zeta\Psi^{>}_m(\xi_P)\right],\\
&\Psi^{>}_{m,\zeta}(\rho)=\frac{1}{2}\left[\Psi^{>}_m(\rho)+\zeta\Psi^{<}_m(\rho_P)\right]=\zeta P_{\rho\xi}\Psi^{<}_{m,\zeta}(\xi)=\zeta\Psi^{<}_{m,\zeta}(\rho_P).
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&\Lambda_m(\mathbf{u})=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{R^2\sqrt{r}}\psi_m(r)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{R^2\sqrt{r}}\psi_{m,+}(r)+\\
&+\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{R^2\sqrt{r}}\psi_{m,-}(r),\ \Psi_{m,\zeta}(r)=\frac{|R^2-r^2|}{R^2\sqrt{r}}\psi_{m,\zeta}(r),\\
&\psi_{m,\zeta}(r)=\left(\begin{array}{c}\psi^{>}_{m,\zeta}(\rho)\\ \psi^{<}_{m,\zeta}(\xi)\end{array}\right),\ \psi^{<}_{m,\zeta}(\xi)=\frac{1}{2}\left[\psi^{<}_m(\xi)+\zeta\frac{\xi_P}{R}\psi^{>}_m(\xi_P)\right],\\
&\psi^{>}_{m,\zeta}(\rho)=\frac{1}{2}\left[\psi^{>}_m(\rho)+\zeta\frac{\rho_P}{R}\psi^{<}_m(\rho_P)\right]=\zeta P_{\rho\xi}\frac{\xi}{R}\psi^{<}_{m,\zeta}(\xi)=\\
&=\zeta\frac{\rho_P}{R}\psi^{<}_{m,\zeta}(\rho_P)=\zeta\psi^{|>}_{\tilde{P}m,\zeta}(\rho),\ \tilde{P}\psi_{m,\zeta}(r)=\zeta\psi_{m,\zeta}(r).
\end{aligned}$$

Below we omit the subscript "ζ" for the lower component of functions $\Psi_{m,\zeta}(r)$ and $\psi_{m,\zeta}(r)$, that is,

$$\begin{aligned}
&\Psi_{m,\zeta}(r)=\left(\begin{array}{c}\Psi^{>}_{m,\zeta}(\rho)\\ \Psi^{<}_m(\xi)\end{array}\right),\ \Psi^{>}_{m,\zeta}(\rho)=\zeta\Psi^{<}_m(\rho_P),\\
&\psi_{m,\zeta}(r)=\left(\begin{array}{c}\psi^{>}_{m,\zeta}(\rho)\\ \psi^{<}_m(\xi)\end{array}\right),\ \psi^{>}_{m,\zeta}(\rho)=\zeta\frac{\rho_P}{R}\psi^{<}_{m,\zeta}(\rho_P),\\
&P\Psi_{m,\zeta}(r)=\zeta\Psi_{m,\zeta}(r),\ \tilde{P}\psi_{m,\zeta}(r)=\zeta\psi_{m,\zeta}(r).
\end{aligned}\tag{2.6}$$

Let $\check{h}(r)\psi_{m,\zeta}(r)\equiv\psi_{h,m,\zeta}(r)$. Then we have

$$\tilde{P}\psi_{h,m,\zeta}(r)=\left(\tilde{P}\check{h}(r)\tilde{P}\right)\left(\tilde{P}\psi_{m,\zeta}(r)\right)=\zeta\check{h}(r)\psi_{m,\zeta}(r)=\zeta\psi_{h,m,\zeta}(r)$$

Note the relation (ζ is fixed)

$$\begin{aligned}
&\int_0^\infty|\psi_{m,\zeta}(r)|^2dr=2\int_0^R|\psi^{<}_m(r)|^2dr,\\
&\int_0^\infty\overline{\psi_{1m,\zeta}(r)}\check{h}(r)\psi_{2m,\zeta}(r)dr=2\int_0^R\overline{\psi^{<}_{1m}(r)}\check{h}(r)\psi^{<}_{2m}(r)dr.
\end{aligned}$$

Let

$$\psi^{(>)}(r)=\left(\begin{array}{c}\psi^{>}(\rho)\\ 0\end{array}\right),\ \psi^{(<)}(r)=\left(\begin{array}{c}0\\ \psi^{<}(\xi)\end{array}\right).$$

Then:

$$\tilde{P}\psi^{(>)}(r)=\left(\begin{array}{c}0\\ \psi_{\tilde{P}}^{|<}(\xi)\end{array}\right),\ \tilde{P}\psi^{(<)}(r)=\left(\begin{array}{c}\psi_{\tilde{P}}^{|>}(\rho)\\ 0\end{array}\right),$$

$$\int_R^\infty \overline{\psi_{1\tilde{P}}^{|>}(\rho)}\psi_{2\tilde{P}}^{|>}(\rho)d\rho=\int_R^\infty \overline{\psi_1^<(\rho_P)}\psi_2^<(\rho_P)\frac{R^2d\rho}{\rho^2}=$$

$$=-\int_R^\infty \overline{\psi_1^<(\rho_P)}\psi_2^<(\rho_P)d\frac{R^2}{\rho}=\int_0^R \overline{\psi_1^<(\rho_P)}\psi_2^<(\rho_P)d\rho_P=\int_0^R \overline{\psi_1^<(\xi)}\psi_2^<(\xi)d\xi,$$

$$\int_0^R \overline{\psi_{1\tilde{P}}^{|<}(\xi)}\psi_{2\tilde{P}}^{|<}(\xi)d\xi=\int_R^\infty \overline{\psi_1^>(\rho)}\psi_2^>(\rho)d\rho,$$

$$\int_0^\infty \overline{\psi_{1\tilde{P}}(r)}\psi_{2\tilde{P}}(r)dr=\int_0^\infty \overline{\psi_1(r)}\psi_2(r)dr.$$

2.2 Reduction to radial problem

In the case under consideration, the Hilbert space $\mathfrak{H}$ is a direct orthogonal sum of subspaces $\mathfrak{H}_{m,\zeta}$, that are the eigenspaces of the rotation operator $\hat{U}_S$ and the parity operator P,

$$\mathfrak{H}=\sum_{m\in\mathbb{Z},\zeta=\pm}^{\oplus}\mathfrak{H}_{m,\zeta},\ \hat{U}_S\mathfrak{H}_{m,\zeta}=e^{-im\theta}\mathfrak{H}_{m,\zeta},\ P\mathfrak{H}_{m,\zeta}=\zeta\mathfrak{H}_{m,\zeta},$$
$$\mathfrak{H}_{m,\zeta}=\hat{P}_{m,\zeta}\mathfrak{H},$$

where θ is the rotation angle corresponding to S, and $\hat{P}_{m,\zeta}$ is an orthohonal projector on subspace $\mathfrak{H}_{m,\zeta}$. $\mathfrak{H}_{m,\zeta}$ consists of eigenfunctions $\Psi_{m,\zeta}(\mathbf{u})$ of angular momentum operator $\hat{L}_z=-i\hbar\partial/\partial\varphi_u$ and parity operator P, $\Psi_{m,\zeta}(\mathbf{u})=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{2R^2\sqrt{r}}\psi_{m,\zeta}(r)$, where

$$\tilde{P}\psi_{m,\zeta}(r)=\zeta\psi_{m,\zeta}(r),$$

and $\psi_{m,\zeta}(r)\in\mathfrak{h}_{m,\zeta}=L^2(\mathbb{R}_+)$, $\mathfrak{h}_{m,\zeta}$ is the Hilbert space of s.-integrable functions on the semi-axis $\mathbb{R}_+$ with scalar product

$$\langle f_1,f_2\rangle=\int_{\mathbb{R}_+}\overline{f_1\left(r\right)}f_2\left(r\right)dr.$$

We define an initial symmetric operator $\hat{H}$ associated with $\check{H}$ as follows:

$$\hat{H}:\left\{\begin{array}{l}D_H=\{\psi(\mathbf{u}):\ \psi\in\mathcal{D}(\mathbb{R}^2\backslash\{r=R\})\}\\ \hat{H}\psi=\check{H}\psi,\ \forall\psi\in D_H\end{array}\right.,$$

where $\mathcal{D}(\mathbb{R}^2\backslash\{r=R\})$ is the space of smooth and compactly supported functions vanishing in a neighborhood of origin and of the circle $r=R$. The domain D_H is dense in $\mathfrak{H}$ and the symmetricity of $\hat{H}$ is obvious. It is also obvious that the operator $\hat{H}$ commutes[1] with the unitary operators $\hat{U}_S$ and the s.a. operator P,

$$\hat{H}=\sum_{m\in\mathbb{Z},,\zeta=\pm}^{\oplus}\hat{H}_{m,\zeta},\ \hat{H}_{m,\zeta}=\hat{P}_{m,\zeta}\hat{H},\ D_{H_{m,\zeta}}=\hat{P}_{m,\zeta}D_H,$$

[1]We remind the reader of the notion of commutativity in this case (where one of the operators, U_S or P, is bounded and defined everywhere): we say that the operators $\hat{H}$ and U_S commute if $U_S\hat{H}\subseteq\hat{H}U_S$, i.e., if $\psi\in D_H$, then also $U_S\psi\in D_H$ and $U_S\hat{H}\psi=\hat{H}U_S\psi$, and analogously for P.

$$\hat{H}\Psi_{m,\zeta}(\mathbf{u}) = \hat{H}_{m,\zeta}\Psi_{m,\zeta}(\mathbf{u}) = \check{H}_m\Psi_{m,\zeta}(\mathbf{u}) =$$
$$= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{R^2\sqrt{r}}\hat{h}_{m,\zeta}\psi_{m,\zeta}(r)\,,$$

where $\hat{h}_{m,\zeta}$ is a symmetric operator defined in the Hilbert space $\mathfrak{h}_{m,\zeta} = L^{2.\zeta}(\mathbb{R}_+)$, $L^{2.\zeta}(\mathbb{R}_+) = \hat{P}_\zeta L^2(\mathbb{R}_+)$, $\hat{P}_\zeta = (1+\zeta\tilde{P})/2$:

$$\hat{h}_{m,\zeta} : \left\{ \begin{array}{l} D_{h_{m,\zeta}} = \mathcal{D}_\zeta(\mathbb{R}_+\backslash\{R\}) \\ \hat{h}_{m,\zeta}\psi_{m,\zeta} = \check{h}_m\psi_{m,\zeta},\ \forall\psi_{m,\zeta}\in D_{h_{m,\zeta}}, \end{array}\right. , \tag{2.7}$$

the differential operation $\check{h}_m$ is given by eq. (2.1) and $\mathcal{D}_\zeta(\mathbb{R}_+\backslash\{R\}) = \hat{P}_\zeta\mathcal{D}(\mathbb{R}_+\backslash\{R\})$.

In what follows, the s.a. operators $\hat{f}$ which commute with the operators $\hat{U}_S$ and P, we will call rotationally- and parity-invariant. Such operators can be represented in the form

$$\hat{f} = \sum\nolimits_{m\in\mathbb{Z},,\zeta=\pm}^{\oplus}\hat{f}_{m,\zeta},\ \hat{f}_{m,\zeta} = \hat{P}_{m,\zeta}\hat{f},$$

and $\hat{f}_{m,\zeta}$ are s.a. operators in sufspaces $\mathfrak{H}_{m,\zeta}$.

Let $\hat{h}_{\mathfrak{e}m,\zeta}$ is a s.a.operator associated with the differential operation $\check{h}_m$ in the Hilbert space $\mathfrak{h}_{m,\zeta}$. Then the operator $\hat{H}_{\mathfrak{e}m,\zeta}$,

$$\hat{H}_{\mathfrak{e}m,\zeta}\Psi_{m,\zeta}(\mathbf{u}) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\varphi_u}\frac{|R^2-r^2|}{R^2\sqrt{r}}\hat{h}_{\mathfrak{e}m,\zeta}\psi_{m,\zeta}(r)\,,\ \psi_{m,\zeta}(r)\in D_{h_{\mathfrak{e}m,\zeta}}, \tag{2.8}$$

is a s.a. operator associated with $\check{H}_m$ in the Hilbert space $\mathfrak{H}_{m,\zeta}$ and operator $\hat{H}_\mathfrak{e}$,

$$\hat{H}_\mathfrak{e} = \sum\nolimits_{m\in\mathbb{Z},\zeta=\pm}^{\oplus}\hat{H}_{\mathfrak{e}m,\zeta}, \tag{2.9}$$

is a s.a. operator in the Hilbert space $\mathfrak{H}$.

Conversely, let $\hat{H}_\mathfrak{e}$ be a rotationally- and parity-invariant s.a. extension of $\hat{H}$. Then it has the form (2.9), where $\hat{H}_{\mathfrak{e}m,\zeta}$ are s.a. operators in $\mathfrak{H}_{m,\zeta}$. The operator $\hat{H}_{\mathfrak{e}m,\zeta}$ acts in subspace $\mathfrak{H}_{m,\zeta}$ by the rule (2.8) with some operator $\hat{h}_{\mathfrak{e}m,\zeta}$ which is obviously a s.a. operator associated with the symmetric operator $\hat{h}_{m,\zeta}$ in the Hilbert space $\mathfrak{h}_{m,\zeta}$.

In what follows, we restrict ourselves to the consideration of the s.a. operators $\hat{H}_\mathfrak{e}$ which are the rotationally- and parity-invariant s.a. extension of $\hat{H}$. As it was explained above, this means that $\hat{H}_\mathfrak{e}$ has a structure of eq. (2.9), acts by the rule (2.8) and $\hat{h}_{\mathfrak{e}m,\zeta}$ is an s.a. extension of the symmetric operator $\hat{h}_m$.

Thus, the problem of constructing a rotationally-invariant s.a. Hamiltonian $\hat{H}_\mathfrak{e}$ is reduced to constructing s.a. radial Hamiltonians $\hat{h}_{\mathfrak{e}m,\zeta}$.

2.3 $|m| \geq 1$

2.3.1 Useful solutions, $r < R$

We need solutions of an equation

$$(\check{h}_{Om} - W_O)\psi^{<}_{Om}(r) = 0, \tag{2.10}$$

where $\check{h}_m$ is given by eq. (2.1) and $W_O = W$ is complex energy,

$$W = |W|e^{i\varphi_W},\ 0 \le \varphi_W \le \pi,\ \operatorname{Im} W \ge 0.$$

It is convenient for our aims first to consider solutions more general equation

$$(\check{h}_{Om,\delta} - W_O)\psi_{m,\delta}(r) = 0,\ \check{h}_{Om,\delta} = \check{h}_{m,\delta} = \check{h}_m\big|_{m\to m+\delta} = \tag{2.11}$$
$$= \frac{\sqrt{r}}{R^2 - r^2}\check{H}_{m,\delta}\frac{R^2 - r^2}{\sqrt{r}},\ \check{H}_{m,\delta} = \check{H}_m\big|_{m\to m+\delta},\ |\delta| < 1;$$
$$(\check{H}_{m,\delta} - W)\Psi_{m,\delta} = 0.\ \Psi_{m,\delta}(r) = \frac{R^2 - r^2}{R^2\sqrt{r}}\psi_{m,\delta}(r). \tag{2.12}$$

Introduce a new variable x,

$$x = \frac{4R^2r^2}{(R^2 + r^2)^2},\ r = R\frac{1 - \sqrt{1 - x}}{\sqrt{x}},\ r\partial_r = 2x\sqrt{1 - x}\partial_x,$$
$$R^2 + r^2 = 2R^2\frac{1 - \sqrt{1 - x}}{x},\ R^2 - r^2 = 2R^2\frac{\sqrt{1 - x}(1 - \sqrt{1 - x})}{x},$$
$$V(r) = \frac{4(q - 1)r^2}{(R^2 + r^2)^2} = \frac{(q - 1)x}{R^2},\ \frac{(R^2 - r^2)^2}{R^4r^2} = \frac{4(1 - x)}{R^2x},$$
$$\Delta_{BLr} = \frac{(R^2 - r^2)^2}{R^4}\frac{1}{r}\partial_r r\partial_r = \frac{16(1 - x)^{3/2}}{R^2}\partial_x x\sqrt{1 - x}\partial_x,$$

and new function $\phi_{\xi_\mu\xi_\nu\delta}(x)$,

$$\psi_{m,\delta}(r) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\xi_\mu\mu_\delta}(1 - x)^{1/4 + \xi_\nu\nu}\phi_{\xi_\mu\xi_\nu\delta}(x),$$
$$\Psi_{m,\delta}(r) = x^{\xi_\mu\mu_\delta}(1 - x)^{1/4 + \xi_\nu\nu}\phi_{\xi_\mu\xi_\nu\delta}(x),$$
$$\mu_\delta = \frac{1}{2}|m + \delta|,\ \nu = \frac{1}{4}\sqrt{q - w},\ w = R^2W,\ \xi_\mu, \xi_\nu = \pm 1,$$

and $\sqrt{q - w}$ is understood as the principal value, real q is understood as the limit $\operatorname{Im} q \to -0$. Note that $\operatorname{Re}\nu > 0$. $\operatorname{Im}\nu < 0$ for $\operatorname{Im} W > 0$.

Then we obtain

$$[x(1 - x)\partial_x^2 + (\gamma_{\xi_\mu\delta} - (1 + \alpha_{\xi_\mu\xi_\nu\delta} + \beta_{\xi_\mu\xi_\nu\delta})x)\partial_x - \alpha_{\xi_\mu\xi_\nu\delta}\beta_{\xi_\mu\xi_\nu\delta}]\phi_{\xi_\mu\xi_\nu\delta}(x) = 0, \tag{2.13}$$
$$\alpha_{\xi_\mu\xi_\nu\delta} = 1/2 + \xi_\mu\mu_\delta + \xi_\nu\nu + \sigma,\ \beta_{\xi_\mu\xi_\nu\delta} = 1/2 + \xi_\mu\mu_\delta + \xi_\nu\nu - \sigma,$$
$$\gamma_{\xi_\mu\delta} = 1 + 2\xi_\mu\mu_\delta,\ \sigma = \left\{\begin{array}{l} \frac{1}{4}\sqrt{q},\ q \ge 0 \\ i\varkappa/4,\ \varkappa = \sqrt{|q|},\ q < 0 \end{array}\right. .$$

Eq. (2.13) is the equation for hypergeometric functions, in the terms of which we can

express solutions of eq. (2.11). We will use the following solutions:

$$
\begin{aligned}
&O^{<}_{1,m,\delta}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^{\mu_\delta}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{1\delta},\beta_{1\delta};\gamma_{1\delta};x) = \\
&\ = Q_{1,m,\delta}(W)O^{<}_{3,m,\delta}(r;W) + Q_{2,m,\delta}(W)v^{<}_{m,\delta}(r;W) = \\
&\ = O^{<}_{1,m,\delta}(r;W)\big|_{\sigma\to-\sigma} = O^{<}_{1,m,\delta}(r;W)\big|_{\nu\to-\nu}\,, \\
&Q_{1,m,\delta}(W) = \frac{\Gamma(\gamma_{1\delta})\Gamma(-2\nu)}{\Gamma(\alpha_{4\delta})\Gamma(\beta_{4\delta})},\ Q_{2,m,\delta}(W) = \frac{\Gamma(\gamma_{1\delta})\Gamma(2\nu)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})} = Q_{1,m,\delta}(W)|_{\nu\to-\nu}\,, \\
&O^{<}_{2,m,\delta}(r;W) = \frac{1}{\Gamma(\gamma_{2\delta})}\frac{R^2\sqrt{r}}{R^2-r^2}x^{-\mu_\delta}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{2\delta},\beta_{2\delta};\gamma_{2\delta};x) = \\
&\ = \frac{1}{\Gamma(\gamma_{2\delta})}\,O^{<}_{1,m,\delta}(r;W)\big|_{\mu_\delta\to-\mu_\delta} = O^{<}_{2,m,\delta}(r;W)\big|_{\sigma\to-\sigma} = O^{<}_{2,m,\delta}(r;W)\big|_{\nu\to-\nu}\,,
\end{aligned}
$$

$$
\begin{aligned}
&(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^{\mu_\delta}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{1\delta},\beta_{1\delta};\gamma_3;1-x), \\
&v^{<}_{m,\delta}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^{\mu_\delta}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_{4\delta},\beta_{4\delta};\gamma_4;1-x) = O^{<}_{3,m,\delta}(r;W)\big|_{\nu\to-\nu}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\alpha_{1\delta,2\delta} = 1/2 \pm \mu_\delta + \nu + \sigma,\ \beta_{1\delta,2\delta} = 1/2 \pm \mu_\delta + \nu - \sigma, \\
&\alpha_{4\delta} = 1/2 + \mu_\delta - \nu + \sigma,\ \beta_{4\delta} = 1/2 + \mu_\delta - \nu - \sigma, \\
&\gamma_{1\delta,2\delta} = 1 \pm 2\mu_\delta,\ \gamma_{3,4} = 1 \pm 2\nu.
\end{aligned}
$$

Note that $O^{<}_{1,m,\delta}(r;W)$ and $O^{<}_{2,m,\delta}(r;W)$ are real-entire in W solutions of eq. (2.11).

Represent $O^{<}_{3,m,\delta}$ in the form

$$
\begin{aligned}
&O^{<}_{3,m,\delta}(r;W) = B_{m,\delta}(W)O^{<}_{1,m,\delta}(r;W) + Q_{3,m,\delta}(W)O^{<}_{4,m,\delta}(r;W), \\
&O^{<}_{4,m,\delta}(r;W) = \Gamma(\gamma_{2\delta})\left[O^{<}_{2,m,\delta}(r;W) - A_{m,\delta}(W)O^{<}_{1,m,\delta}(r;W)\right], \\
&Q_{3,m,\delta}(W) = \frac{\Gamma(\gamma_3)\Gamma(2\mu_\delta)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})},\ A_{m,\delta}(W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_{1\delta})}, \\
&B_{m,\delta}(W) = \frac{\Gamma(\gamma_3)\Gamma(-2\mu_\delta)}{\Gamma(\alpha_{2\delta})\Gamma(\beta_{2\delta})} + \frac{\Gamma(\gamma_3)\Gamma(2\mu_\delta)\Gamma(\gamma_{2\delta})}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}A_{m,\delta}(W) = \\
&\ = \frac{\Gamma(\gamma_3)\Gamma(-2\mu_\delta)}{\Gamma(\alpha_2)\Gamma(\beta_2)}\left[\frac{\Gamma(\alpha_2)\Gamma(\beta_2)}{\Gamma(\alpha_{2\delta})\Gamma(\beta_{2\delta})} - \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}\right], \\
&\alpha_{1,2} = 1/2 \pm \mu + \nu + \sigma,\ \beta_{1,2} = 1/2 \pm \mu + \nu - \sigma,\ \mu = |m|/2,
\end{aligned}
$$

where we used eq.9.131.2 in [6]. The function $A_{m,\delta}(W)$ can be represented in the form

$$A_{m,\delta}(W) = \frac{1}{\Gamma(\gamma_{1\delta})}\mathcal{A}_m(W),$$
$$\mathcal{A}_m(W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)} = \prod_{k=1}^{|m|}(\alpha_1 - k)(\beta_1 - k) =$$
$$= \frac{1}{\Gamma(\gamma_{1\delta})}\left\{ \begin{array}{c} (v^2-\sigma^2)I_p(\nu,\sigma^2)I_p(-\nu,\sigma^2),\ |m| = 2p-1 \\ J_p(\nu,\sigma^2)J_p(-\nu,\sigma^2),\ |m| = 2p \end{array} \right. ,$$
$$p = 1,2,...,\ I_1(z) = 1,\ I_p(z,\sigma^2) = \prod_{k=1}^{p-1}\left[(k+z)^2 - \sigma^2\right],\ p \geq 2,$$
$$J_p(z) = \prod_{k=0}^{p-1}\left[(k+1/2+z)^2 - \sigma^2\right],$$

i.e., $A_{m,\delta}(W)$ is a polynomial in W with real coefficients, and, therefore, $O^<_{4,m,\delta}(r;W)$ is real-entire in W.

We obtain the solution of eq.(2.10) as the limit $\delta \to 0$ of the solution of eq. (2.11):

$$O^<_{1,m}(r;W) = O^<_{1,m,0}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^\mu(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1,\beta_1;\gamma_1;x) =$$
$$(\text{for} = \operatorname{Im} W > 0) = Q_1(W)O^<_{3,m}(r;W) + Q_2(W)v^<_m(r;W),\ \gamma_1 = 1+2\mu,$$
$$Q_1(W) = Q_{10}(W) = \frac{\Gamma(\gamma_1)\Gamma(-2\nu)}{\Gamma(\alpha_4)\Gamma(\beta_4)},\ Q_2(W) = Q_{20}(W) = \frac{\Gamma(\gamma_1)\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)},$$
$$v^<_m(r;W) = v^<_{m,0}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^\mu(1-x)^{1/4-\nu}\mathcal{F}(\alpha_4,\beta_4;\gamma_4;1-x),$$
$$O^<_{4,m}(r;W) = \lim_{\delta\to 0} O^<_{4,m,\delta}(r;W),$$
$$O^<_{3,m}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}x^\mu(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1,\beta_1;\gamma_3;1-x) =$$
$$= B_m(W)O^<_{1,m}(r;W) + C_m(W)O^<_{4,m}(r;W),\ C_m(W) = \frac{\Gamma(\gamma_3)\Gamma(|m|)}{\Gamma(\alpha_1)\Gamma(\beta_1)},$$
$$B_m(W) = B_{m,0}(W) = \frac{(-1)^{|m|+1}\Gamma(\gamma_3)}{2\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_1)}\left[\psi(\alpha_1)+\psi(\alpha_2)+\psi(\beta_1)+\psi(\beta_2)\right],$$

where we used relations.

$$|m+\delta| = |m| + \delta_m,\ \delta_m = \delta\operatorname{sign}m,\ \Gamma(-|m+\delta|) = \frac{(-1)^{|m|+1}}{\Gamma(\gamma_1)}\frac{1}{\delta_m},$$
$$O^<_{2,m,0}(r;W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_1)}O^<_{1,m}(r;W)$$
$$\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_{1,\delta})} = 1 - \frac{1}{2}\delta_m\psi(\alpha_1),\ \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_{2,\delta})} = 1 + \frac{1}{2}\delta_m\psi(\alpha_2),$$
$$\frac{\Gamma(\beta_1)}{\Gamma(\beta_{1,\delta})} = 1 - \frac{1}{2}\delta_m\psi(\beta_1),\ \frac{\Gamma(\beta_2)}{\Gamma(\beta_{2,\delta})} = 1 + \frac{1}{2}\delta_m\psi(\beta_2).$$

Note that $O^<_{1,m}(r;W)$ and $O^<_{4,m}(r;W)$ are real-entire in W.

2.3.2 Asymptotics, $r \to 0$ $(x \to 0)$

We have

$$
\begin{aligned}
&x = \frac{4r^2}{R^2}(1+O(r^2)),\ p_0(r) = 1+O(r^2),\\
&O^<_{1,m}(r;W) = (2/R)^{|m|} r^{1/2+|m|}(1+O(r^2)),\\
&O^<_{4,m}(r;W) = (R/2)^{|m|} r^{1/2-|m|}\left(1+\left\{\begin{array}{c} O(r^2),\ |m|\geq 2 \\ O(r^2\ln r),\ |m|=1 \end{array}\right.\right),
\end{aligned}
$$

$$
\begin{aligned}
&O^<_{3,m}(r;W) = .\frac{\Gamma(\gamma_3)\Gamma(2\mu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}(R/2)^{|m|} r^{1/2-|m|}\left(1+\left\{\begin{array}{c} O(r^2),\ |m|\geq 2 \\ O(r^2\ln r),\ |m|=1 \end{array}\right.\right),\\
&\operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

2.3.3 Asymptotics, $\Delta = R - r \to 0$ $(\delta = 1 - x \to 0)$

We have

$$
\begin{aligned}
&\delta = \frac{\Delta^2}{R^2}(1+O(\Delta)),\ p_0(r) = \frac{4\Delta^2}{R^2}(1+O(\Delta)),\\
&O^<_{3,m}(r;W) = \frac{1}{2}R^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),
\end{aligned}
$$

$$
\begin{aligned}
&O^<_{1,m}(r;W) = \frac{\Gamma(\gamma_1)\Gamma(2\nu)}{2\Gamma(\alpha_1)\Gamma(\beta_1)}R^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)),\\
&\operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

2.3.4 Wronskian

The Wronskian $\mathrm{Wr}(U,V)$ of two functions $U(r)$ and $V(r)$ is equal to

$$
\mathrm{Wr}(U,V) = p_0(r)[U(r)\partial_r V(r) - V(r)\partial_r U(r)].
$$

We have

$$
\mathrm{Wr}(O^<_{1,m}, O^<_{3,m}) = -2\frac{\Gamma(\gamma_1)\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)} = -\omega_m(W) = -2|m|C_m(W).
$$

2.3.5 Useful solutions, $r > R$

We need solutions of eq. (2.10) for $r > R$. As two independent solutions, we choose the following functions:

$$
O^>_{1,m}(r;W) = \frac{R}{r}O^<_{1,m}(R^2/r;W), \tag{2.14}
$$

$$
O^>_{3,m}(r;W) = \frac{R}{r}O^<_{3,m}(R^2/r;W) \tag{2.15}
$$

According to eq. (2.5), the functions (2.14) and (2.15) satisfy really eq. (2.10) for $r > R$.

2.3.6 Asymptotics, $\Delta = r - R \to 0$

We have

$$O^{>}_{3,m}(r;W) = \frac{1}{2}R^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),\ \Delta = R - r,$$

$$O^{<}_{1,m}(r;W) = \frac{\Gamma(\gamma_1)\Gamma(2\nu)}{2\Gamma(\alpha_1)\Gamma(\beta_1)}R^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)),$$
$$\operatorname{Im} W > 0 \text{ or } W = 0.$$

2.3.7 Asymptotics, $r \to \infty$

We have

$$O^{>}_{1,m}(r;W) = 2^{|m|}R^{2+|m|}r^{-3/2-|m|}(1+O(r^{-2})),$$

$$O^{>}_{3,m}(r;W) = \frac{\Gamma(\gamma_3)\Gamma(2\mu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}2^{-|m|}R^{2-|m|}r^{-3/2+|m|}\left(1+\left\{\begin{array}{c} O(r^{-2}),\ |m|\geq 2 \\ O(r^{-2}\ln r),\ |m| = 1\end{array}\right.\right),$$
$$\operatorname{Im} W > 0 \text{ or } W = 0.$$

We see that eq. (2.10) has no s.-integrable silutions for $\operatorname{Im} W > 0$. This means that the deficiency indices of the symmetric operator (see below) are equal to $(0,0)$.

2.3.8 Symmetric operator $\hat{h}_{m,\zeta}$

For given a differential operation $\check{h}_m$, the symmetric operator $\hat{h}_{m,\zeta}$ is given by eq. 2.7.

2.3.9 Isometry

We remind that the functions $\psi_{m,\zeta}(r) \in \mathfrak{h}_{m,\zeta} = L^2(\mathbb{R}_+)$ have the structure, see eqs. (2.6), (2.3) and (2.4),

$$\psi_{m,\zeta}(r) = \left(\begin{array}{c} \psi^{>}_{m,\zeta}(\rho) \\ \psi^{<}_{m}(\xi)\end{array}\right),\ \psi^{>}_{m,\zeta}(\rho) = \zeta\frac{\rho_P}{R}\psi^{<}_{m}(\rho_P) = \zeta P_{\rho\xi}\frac{\xi}{R}\psi^{<}_{m}(\xi),$$

$$\hat{h}_{m,\zeta}(r)\psi_{m,\zeta}(r) = \check{h}_m(r)\psi_{m,\zeta}(r) = \left(\begin{array}{c} \check{h}^{>}_{m}(\rho)\psi^{>}_{m,\zeta}(\rho) \\ \check{h}^{<}_{m}(\xi)\psi^{<}_{m}(\xi)\end{array}\right) =$$
$$= \left(\begin{array}{c} \zeta P_{\rho\xi}\frac{\xi}{R}\check{h}^{<}_{m}(\xi)\psi^{<}_{m}(\xi) \\ \check{h}^{<}_{m}(\xi)\psi^{<}_{m}(\xi)\end{array}\right),\ \psi^{<}_{m}(\xi) \in \mathcal{D}(0,R).$$

Let us untroduce an isometric map T (ζ is fixed),

$$\psi_{m,\zeta}(r) \in \mathfrak{h}_{m,\zeta} = L^{2,\zeta}(\mathbb{R}_+) \overset{T}{\leftrightarrow} \psi_{T,m,\zeta}(\xi) = \sqrt{2}\psi^{<}_{m}(\xi) \in \mathfrak{h}^{(1/2)}_{m} = L^2(0,R),$$

$$D_{h_{m,\zeta}} = \mathcal{D}_\zeta(\mathbb{R}_+\backslash\{R\}) \overset{T}{\leftrightarrow} D_{h^{(1/2)}_m} = \mathcal{D}(0,R)$$

$$\hat{h}_{m,\zeta}(r)\psi_{m,\zeta}(r) = \begin{pmatrix} \zeta P_{\rho\xi}\frac{\xi}{R}\check{h}_m^<(\xi)\psi_{m,\zeta}^<(\xi) \\ \check{h}_m^<(\xi)\psi_{m,\zeta}^<(\xi) \end{pmatrix} \overset{T}{\leftrightarrow} \hat{h}_{m,\zeta}^{(1/2)}(\xi)\psi_{T,m,\zeta}(\xi) =$$
$$= \sqrt{2}\check{h}_m^<(\xi)\psi_m^<(\xi),\ \hat{h}_{m,\zeta}^{(1/2)} = \hat{h}_m^{(1/2)} = \hat{h}_m^<(\xi),\ \psi_{m,\zeta} \in D_{h_{m,\zeta}},\ \psi_{T,m,\zeta} \in D_{h_{m,\zeta}^{(1/2)}}.$$

Let $\hat{h}_{\mathfrak{e}m,\zeta}^{(1/2)}$ be an s.a. extension of the symmetric operator $\hat{h}_m^{(1/2)}$. Then the corresponding s.a. operator $\hat{H}_{\mathfrak{e}m,\zeta}$, an s.a. extension of symmetric operator $\hat{H}_{m,\zeta}$, can be reconstructed by the rule (2.8) with $\hat{h}_{\mathfrak{e}m,\zeta}$ be given by the expression (see eq. (2.3)),

$$\hat{h}_{\mathfrak{e}m,\zeta}(r) = .\begin{pmatrix} \hat{h}_{\mathfrak{e}m,\zeta}^>(\rho) & 0 \\ 0 & \hat{h}_{\mathfrak{e}m,\zeta}^{(1/2)}(\xi) \end{pmatrix},\ \hat{h}_{\mathfrak{e}m,\zeta}^>(\rho) = \frac{R}{\rho}\hat{h}_{\mathfrak{e}m,\zeta}^{(1/2)}(\rho_P)\frac{\rho}{R}.$$

Pass to the construction of s.a. operators $\hat{h}_{\mathfrak{e}m,\zeta}^{(1/2)}$ which are s.a. extensions of the symmetric operators $\hat{h}_{m,\zeta}^{(1/2)} = \hat{h}_{m,\zeta}^<(\xi)$ associated with the differential operations $\check{h}_m^<(\xi)$ in the Hilbert space $\mathfrak{h}_{m,\zeta}^{(1/2)} = L^2(0,R)$.

2.3.10 Adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$

We will omit the superscript (1/2).and will write r for ξ, $0 \le r \le R$.

It is easy to prove by standard way that the adjoint operator $\hat{h}_m^+$ coincides with the operator $\hat{h}_m^*$,

$$\hat{h}_m^+ : \begin{cases} D_{h_m^+} = D_{\check{h}_m}^*(0,R) = \{\psi_*, \psi_*' \text{ are a.c. in } (0,R),\ \psi_*, \hat{h}_m^+\psi_* \in L^2(0,R)\} \\ \hat{h}_m^+\psi_*(r) = \check{h}_m\psi_*(r),\ \forall\psi_* \in D_{h_m^+} \end{cases}.$$

2.3.11 Asymptotics

Because $\check{h}_m\psi_* \in L^2(0,R)$, we have

$$\check{h}_m\psi_*(r) = \eta(r),\ \eta \in L^2(0,R),$$

and we can represent ψ_* in the form

$$\psi_*(r) = c_1 O_{1,m}^<(r;0) + c_2 O_{3,m}^<(r;0) + I(r),$$
$$\psi_*'(r) = c_1\partial_r O_{1,m}^<(r;0) + c_2\partial_r O_{3,m}^<(r;0) + I'(r),$$

where

$$I(r) = \frac{O_{1,m}^<(r;0)}{\omega_m(0)}\int_r^R O_{3,m}^<(y;0)\eta(y)dy + \frac{O_{3,m}^<(r;0)}{\omega_m(0)}\int_0^r O_{1,m}^<(y;0)\eta(y)dy,$$
$$I'(r) = \frac{\partial_r O_{1,m}^<(r;0)}{\omega_m(0)}\int_r^R O_{3,m}^<(y;0)\eta(y)dy + \frac{\partial_r O_{3,m}^<(r;0)}{\omega_m(0)}\int_0^r O_{1,m}^<(y;0)\eta(y)dy.$$

I) $r \to 0$

We obtain with the help of the Cauchy-Bunyakovskii inequality (CB-inequality):

$$I(r) = \begin{cases} O(r^{3/2}),\ |m| \ge 2 \\ O(r^{3/2}\sqrt{\ln r}),\ |m| = 1 \end{cases},\ I'(r) = \begin{cases} O(r^{1/2}),\ |m| \ge 2 \\ O(r^{1/2}\sqrt{\ln r}),\ |m| = 1 \end{cases},$$

such that we have

$$\psi_*(r) = c_2 c r^{1/2-|m|}\left(1+\left\{\begin{array}{c} O(r^2),\ |m|\geq 2 \\ O(r^2\ln r),\ |m|=1 \end{array}\right.\right)+$$
$$+\left\{\begin{array}{c} O(r^{3/2}),\ |m|\geq 2 \\ O(r^{3/2}\sqrt{\ln r}),\ |m|=1 \end{array}\right.,\ c=(R/2)^{|m|}\frac{\Gamma(\gamma_3)\Gamma(2\mu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}.$$

The condition $\psi_* \in L^2(0,R)$ gives $c_2 = 0$, such that we find finally

$$\psi_*(r) = \left\{\begin{array}{c} O(r^{3/2}),\ |m|\geq 2 \\ O(r^{3/2}\sqrt{\ln r}),\ |m|=1 \end{array}\right.,\ \psi_*'(r) = \left\{\begin{array}{c} O(r^{1/2}),\ |m|\geq 2 \\ O(r^{1/2}\sqrt{\ln r}),\ |m|=1 \end{array}\right.,$$
$$[\psi_*,\chi_*]_0 = 0,\ \forall\psi_*,\chi_* \in D_{h_m^+}. \tag{2.16}$$

II) $r \to R$

In this case, we prove that $[\psi_*,\chi_*]^R = 0,\ \forall\psi_*,\chi_* \in D_{h_m^+}$. Indeed, consider the Hilbert space $\mathfrak{h}_{c,m} = L^2(c,R)$, c is an interior point of the interval $(0,R)$. and an symmetric operator $\hat{h}_{c,m}$, $D_{h_{c,m}} = \mathcal{D}(c,R)$, acting as $\check{h}_m$. We choose the functions $O_{1,m}^<(r;W)$ and $O_{3,m}^<(r;W)$ as the independent solutions of eq. (2.10) for $\operatorname{Im} W > 0$. The left end c of the interval $(0,R)$ is regular and both solutions $O_{1,m}^<$ and $O_{3,m}^<$ are s.-interable on the end c. The right end R is singular. On the right end R, the solution $O_{3,m}^<$ is s.-integrable, but $O_{1,m}^<$ is not. Thus, there is only one s. integrable solution of eq. (2.10) on the interval (c,R) for $\operatorname{Im} W > 0$ and the deficient indexes of the symmetric operator $\hat{h}_{c,m}$ are equal to $(1,1)$. In this case, according to [[7], Lemma on the page 213], we have $[\psi_*,\chi_*]^R = 0,\ \forall\psi_*,\chi_* \in D_{h_{c,m}^+}$. Because the restriction ψ_{c*} on the interval (c,R) of any function $\psi_* \in D_{h_m^+}$ belongs to $D_{h_{c,m}^+}$, $\psi_{c*} \in D_{h_{c,m}^+}$. $\forall\psi_* \in D_{h_m^+}$, we obtain that $[\psi_*,\chi_*]^R = 0,\ \forall\psi_*,\chi_* \in D_{h_m^+}$.

2.3.12 Self-adjoint hamiltonian $\hat{h}_{\mathfrak{e}m,\zeta}$

Because $\omega_{h_m^+}(\chi_*,\psi_*) = \Delta_{h_m^+}(\psi_*) = 0$ (and also because $O_{1,m}^<(r;W)$ and $O_{3,r}^<(u;W)$ and any their linear combinations are not s.-integrable on the interval $(0,R)$ for $\operatorname{Im} W \neq 0$), the deficiency indices of initial symmetric operator $\hat{h}_m$ are zero, which means that $\hat{h}_{\mathfrak{e}m,\zeta} = \hat{h}_{\mathfrak{e}m} = \hat{h}_m^+$ is a unique s.a. extension of the initial symmetric operator $\hat{h}_m$:

$$\hat{h}_{\mathfrak{e}m} : \left\{\begin{array}{l} D_{h_{\mathfrak{e}m}} = D^*_{\check{h}_m}(0,R) \\ \hat{h}_{\mathfrak{e}m}\psi_*(r) = \check{h}_m\psi_*(r),\ \forall\psi_* \in D_{h_{\mathfrak{e}m}} \end{array}\right..$$

2.3.13 The guiding functional $\Phi(\xi;W)$

As a guiding functional (see [8, 7])$\Phi(\xi;W)$ we choose

$$\Phi(\xi;W) = \int_0^R O_{1,m}^<(r;W)\xi(r)dr,\ \xi \in \mathbb{D} = D_r(0,R)\cap D_{h_{\mathfrak{e}m}}. \tag{2.17}$$
$$D_r(0,R) = \{\xi(u) :\ \operatorname{supp}\xi \subseteq [0,\beta_\xi],\ \beta_\xi < R\}.$$

The guiding functional $\Phi(\xi;W)$ is simple. It has, obviously, the properties 1) and 3) and we should prove the properties 2) only. (see [7], pages 245-246). Let $\Phi(\xi_0;E_0) = 0$, $\xi_0 \in \mathbb{D}$, $E_0 \in \mathbb{R}$. As a solution $\psi(r)$ of equation

$$(\check{h}_m - E_0)\psi(r) = \xi_0(r),$$

we choose

$$\psi(r)=O^{<}_{1,m}(r;E_0)\int_r^R U(r)\xi_0(r)dr+U(r)\int_0^r O^{<}_{1,m}(r;E_0)\xi_0(r)dr,$$

where $U(r)$ is any solution of eq. $(\check{h}_m-E_0)U(r)=0$ satisfying the condition $\mathrm{Wr}(O^{<}_{1,m},U)=-1$. Because $\xi_0\in D_r$, the function $\psi(r)$ is well determined. Because $\xi_0\in D_r$ and $\int_0^r O^{<}_{1,m}(r;E_0)\xi_0(r)=0$ for $r>\beta_{\xi_0}$, we have $\psi(r)=0$ for $u>\beta_{\xi_0}$. Using the CB-inequality we show that $\psi(r)$ satisfies the boundary condition (2.16), that is, $\psi\in\mathbb{D}$. Thus, the guiding functional $\Phi(\xi;W)$ is simple and the spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple.

2.3.14 Green function $G_m(r,y;W)$, spectral function $\sigma_m(E)$

We find the Green function $G_m(r,y;W)$ as the kernel of the integral representation

$$\psi(r)=\int_0^R G_m(r,y;W)\eta(y)dy,\ \eta\in L^2(0,R),$$

of unique solution of an equation

$$(\hat{h}_{\mathfrak{e}m}-W)\psi(r)=\eta(r),\ \mathrm{Im}\,W>0, \tag{2.18}$$

for $\psi\in D_{h_{\mathfrak{e}m}}$. General solution of eq. (2.18) can be represented in the form

$$\begin{aligned}
&\psi(r)=a_1O^{<}_{1,m}(r;W)+a_3O^{<}_{3,m}(r;W)+I(r),\\
&I(r)=\frac{O^{<}_{1,m}(r;W)}{\omega_m(W)}\int_r^R O^{<}_{3,m}(y;W)\eta(y)dy+\frac{O^{<}_{3,m}(r;W)}{\omega_m(W)}\int_0^r O^{<}_{1,m}(y;W)\eta(y)dy,\\
&I(r)=\left\{\begin{array}{c} O(r^{3/2}),\ |m|\geq 2\\ O(r^{3/2}\sqrt{\ln r}),\ |m|=1\end{array}\right.,\ r\to 0,\ I(r)=O\left(\Delta^{-1/2}\right),\ r\to R.
\end{aligned}$$

A condition $\psi\in L^2(0,R)$ gives $a_1=a_3=0$, such that we find

$$\begin{aligned}
&G_m(r,y;W)=\frac{1}{\omega_m(W)}\left\{\begin{array}{l} O^{<}_{3,m}(r;W)O^{<}_{1,m}(y;W),\ r>y\\ O^{<}_{1,m}(r;W)O^{<}_{3,m}(y;W).\ r<y\end{array}\right.=\\
&=\pi\Omega_m(W)O^{<}_{1,m}(r;W)O^{<}_{1,m}(y;W)+\frac{1}{2|m|}\left\{\begin{array}{l} O^{<}_{4,m}(r;W)O^{<}_{1,m}(y;W),\ r>y\\ O^{<}_{1,m}(r;W)O^{<}_{4,m}(y;W),\ r<y\end{array}\right.,\\
&\Omega_m(W)\equiv\frac{B_m(W)}{\pi\omega_m(W)}=\frac{(-1)^{|m|+1}[\psi(\alpha_1)+\psi(\alpha_2)+\psi(\beta_1)+\psi(\beta_2)]\mathcal{A}_m(W)}{4\pi\Gamma^2(\gamma_1)}.
\end{aligned} \tag{2.19}$$

Note that the last term in the r.h.s. of eq. (2.19) is real for $W=E$. From the relation

$$[O^{<}_{1,m}(r_0;E)]^2\sigma'_m(E)=\frac{1}{\pi}\,\mathrm{Im}\,G_m(r_0-0,r_0+0;E+i0),$$

where $f(E+i0)\equiv\lim_{\varepsilon\to+0}f(E+i\varepsilon)$, $\forall f(W)$, we find

$$\sigma'_m(E)=\mathrm{Im}\,\Omega_m(E+i0). \tag{2.20}$$

Consider $\Omega_m(W)$ in more details.

Using relations

$$\psi(\alpha_2)=\psi(\alpha_1)-T_{(\alpha)},\ T_{(\alpha)}=\sum_{k=1}^{|m|}\frac{1}{\alpha_1-k}=$$
$$=\sum_{k=1}^{|m|}\frac{1}{\left(\frac{1}{2}+\frac{|m|}{2}+\nu-k\right)+\sigma}=\sum_{k=1}^{|m|}\frac{1}{\nu+\left(\frac{1}{2}+\frac{|m|}{2}+\sigma-k\right)},$$

$$\psi(\beta_2)=\psi(\beta_1)-T_{(\beta)},\ T_{(\beta)}=\sum_{k=1}^{|m|}\frac{1}{\beta_1-k}=$$
$$=\sum_{k=1}^{|m|}\frac{1}{\left(\frac{1}{2}+\frac{|m|}{2}+\nu-k\right)-\sigma}=\sum_{k=1}^{|m|}\frac{1}{\nu-\left(\frac{1}{2}+\frac{|m|}{2}+\sigma-k\right)},$$

$$T_{(\alpha,\beta)}(W)=T_{(\alpha)}+T_{(\beta)}=\sum_{k=1}^{|m|}\frac{1+|m|+2\nu-2k}{\left(\frac{1}{2}+\frac{|m|}{2}+\nu-k\right)^2-\sigma^2}=$$
$$=2\nu\sum_{k=1}^{|m|}\frac{1}{\nu^2-\left(\frac{1}{2}+\frac{|m|}{2}+\sigma-k\right)^2}\Longrightarrow$$
$$\Longrightarrow T_{(\alpha,\beta)}(W)=\frac{2\nu\mathcal{B}_m(W)}{\mathcal{A}_m(W)},$$

where $\mathcal{B}_m(W)$ is an polinomoal in ν and σ, even in both ν and σ, and therefore, is a real-entire polynomial in W, we can represent $\Omega_m(W)$ in the form

$$\Omega_m(W)=\Omega_{1m}(W)+\Omega_{2m}(W),$$
$$\Omega_{1m}(W)=\frac{(-1)^{|m|+1}\mathcal{A}_m(W)}{2\pi\Gamma^2(\gamma_1)}\left[\psi(\alpha_1)+\psi(\beta_1)\right],$$
$$\Omega_{2m}(W)=-\nu\tilde{\Omega}_{2m}(W),\ \tilde{\Omega}_{2m}(W)=\frac{(-1)^{|m|+1}\mathcal{B}_m(W)}{2\pi\Gamma^2(\gamma_1)}.$$

2.3.15 Spectrum

2.3.16 $w=R^2E>q$

In this case, we have $\alpha_1=1/2+|m|/2+\sigma-i\sqrt{w-q}/4$, $\beta_1=1/2+|m|/2-\sigma-i\sqrt{w-q}/4$ and it is easy to prove that $\alpha_1,\beta_1\notin\mathbb{Z}_-$, such that $\Omega_m(E)$ is finite complex function of E and we have

$$\sigma_m'(E)=\operatorname{Im}\Omega_m(E)\equiv\varrho_m^2(E)>0. \tag{2.21}$$

The spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple and continuous, $\operatorname{spec}\hat{h}_{\mathfrak{e}m}=[q/R^2,\infty)$. Note that

$$\lim_{\Delta\to+0}\varrho_m(E)=\left\{\begin{array}{c}0,\ q\neq q_{m,k}\\ O(\Delta^{1/4}),\ q=q_{m,k},\end{array}\right., \tag{2.22}$$

$\Delta=E-q/R^2\to+0$, $q_{m,k}=4N_{m,k}^2$, $N_{m,k}=1+|m|+2k$.

2.3.17 $w = R^2E \le q$

$q > 0$**,** $\sigma > 0$ In this case, we have $\operatorname{Im}\nu|_{W=E} = \operatorname{Im}\alpha_1|_{W=E} = \operatorname{Im}\beta_1|_{W=E} = 0$,
$\operatorname{Im}\psi(\alpha_1)|_{W=E} = 0$, and

$$\sigma_m'(E) = \frac{(-1)^{|m|+1}\mathcal{A}_m(E)}{2\pi\Gamma^2(\gamma_1)}\operatorname{Im}\psi(\beta_1)|_{W=E+i0}\,.$$

$\sigma_m'(E)$ can be different from zero in the points $E_{m,n}$ when $\beta_1 = -n$, $n = 0, 1, 2, ...$, i.e., $E_{m,n}$ satisfy the equations

$$\sqrt{q - w_{m,n}} = \sqrt{q} - 2N_{m,n},\ N_{m,n} = 1 + |m| + 2n \implies \tag{2.23}$$
$$\implies E_{m,n} = \frac{q}{R^2} - \frac{\left(\sqrt{q} - 2N_{m,n}\right)^2}{R^2} = \frac{4\sqrt{q}N_{m,n}}{R^2} - \frac{4N_{m,n}^2}{R^2}.$$

Eq. (2.23) has solutions only if $q \ge q_{m,0}$, $q_{m,0} = 4N_{m,0}^2 = 4(1 + |m|)^2$. With property (2.23) taken into account, we have: $n = 0, 1, ..., n_{\max}$, $E_{m,n} > E_{m,n-1}$, $n = 1, ..., n_{\max}$, $0 < E_{m,n} \le q/R^2$, where

$$\left\{\begin{array}{l} q \le q_{m,0},\ \text{no levels} \\ n_{\max} = k,\ \sqrt{q} = 2(1 + |m| + 2(k + \delta) \end{array}\right.,$$
$$k = 0.1, ...,\ 0 < \delta \le 1,$$

$$\sigma_m'(E) = \sum_{n=0}^{n_{\max}} Q_{m,n}^2\delta(E - E_{m,n}),\ Q_{m,n} = \frac{2\sqrt{(-1)^{|m|}\mathcal{A}_m(E_{m,n})\sqrt{q - w_{m,n}}}}{R\Gamma(\gamma_1)}..$$

We note that the nonexistence of the level $E_{m,n_{\max}+1} = E_{m,k+1} = q/R^2$ for $q = q_{m,k+1}$.follows from the fact that $Q_{m,k+1} = 0$. Thus, the discrete part of the spectrum of $\hat{h}_{\mathfrak{c}m}$ is simple and has the form

$$\operatorname{spec}\hat{h}_{\mathfrak{c}m} = \{E_{m,n},\ 0 < E_{m,n} < q/R^2,\ n = 0, 1, ..., n_{\max}\},$$
$$n_{\max} = k \text{ for } \frac{1}{4}\sqrt{q} = \frac{1 + |m|}{2} + k + \delta,\ 0 < \delta \le 1,\ k \in \mathbb{Z}_+$$

The discrete part of the spectrum is absent for $q \le q_{m,0}$.

$q = 0$**,** $\sigma = 0$ We have in this case for $W = E$: $\alpha_1 = \beta_1 = 1/2 + |m| + \sqrt{|w|}$, $\operatorname{Im}\nu = 0$, $\operatorname{Im}\alpha_1 = 0$, $\alpha_1 > 0$, , and $\sigma_m'(E) = 0$, the spectrum points are absent.

$q < 0$**,** $\sigma = i\varkappa$**,** $\varkappa = \sqrt{|q|}$ In this case, we have for $W = E$: $\nu = \sqrt{|w| - |q|} > 0$, $\alpha_1 = 1/2 + |m|/2 + \nu + i\varkappa$, $\beta_1 = 1/2 + |m|/2 + \nu - i\varkappa = \overline{\alpha_1}$, such that $[\operatorname{Im}\psi(\alpha_1) + \operatorname{Im}\psi(\beta_1)]_{W=E} = 0$, and

$$\sigma_m'(E) = 0.$$

Finally, we find for fixed m, $|m| \ge 1$:

The spectrum of $\hat{h}_{\mathfrak{c}m}$ is simple, $\operatorname{spec}\hat{h}_{\mathfrak{c}m} = [q/R^2, \infty) \cup \{E_{m,n},\ n = 0, 1, ...n_{\max}\}$, the discrete part of spectrum is present for $q > q_{m,0}$. The set of functions

$$\left\{U_m(r; E) = \varrho_m(E)O_{1m}^<(r; E),\ E \ge q/R^2;\ U_{m,n}(r) = Q_{m,n}O_{1m}^<(r; E_{m,n}),\ n = 0, 1, ...n_{\max}\right\}$$

forms a complete orthogonalized system in $L^2(0, R)$.

2.4 $m = 0$

In this case, we have: $\mu_\delta = \delta/2$, $\delta \geq 0$; $\mu = 0$; $\alpha_1 = 1/2 + \nu + \sigma$, $\beta_1 = 1/2 + \nu - \sigma$.

2.4.1 Useful solutions, $r < R$

We need solutions of an equation

$$(\check{h}_0 - W)\psi_0^<(r) = 0, \tag{2.24}$$

where $\check{h}_0$ is given by eq. (2.1).

It is convenient for our aims first to consider solutions more general equation

$$(\check{h}_{0,\delta} - W)\psi_\delta(r) = 0,\ \check{h}_{0,\delta} = \check{h}_m\big|_{m\to\delta}\,. \tag{2.25}$$

Introduce a new function $\phi_{\xi_\mu\delta}(x)$,

$$\psi_\delta(r) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\xi_\mu\delta/2}(1-x)^{1/4+\nu}\phi_{\xi_\mu\delta}(x).$$

Then we obtain

$$\begin{aligned}
&[x(1-x)\partial_x^2 + (\gamma_{\xi_\mu\delta} - (1 + \alpha_{\xi_\mu\delta} + \beta_{\xi_\mu\delta})x)\partial_x - \alpha_{\xi_\mu\delta}\beta_{\xi_\mu\delta}]\phi_{\xi_\mu\delta}(x) = 0, \\
&\alpha_{\xi_\mu\delta} = 1/2 + \xi_\mu\delta/2 + \nu + \sigma,\ \beta_{\xi_\mu\delta} = 1/2 + \xi_\mu\delta/2 + \nu - \sigma,\ \gamma_{\xi_\mu\delta} = 1 + \xi_\mu\delta.
\end{aligned} \tag{2.26}$$

Eq. (2.26) is the equation for hypergeometric functions, in the terms of which we can express solutions of eq. (2.25). We will use the following solutions:

$$\begin{aligned}
&O_{1,0,\delta}^<(r;W) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\delta/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_{1\delta}; x) = \\
&= \frac{\Gamma(\gamma_{1\delta})\Gamma(-2\nu)}{\Gamma(\alpha_{4\delta})\Gamma(\beta_{4\delta})}O_{3,0,\delta}^<(r;W) + \frac{\Gamma(\gamma_{1\delta})\Gamma(2\nu)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}v_{0,\delta}^<(r;W) = \\
&= O_{1,0,\delta}^<(r;W)\big|_{\sigma\to-\sigma} = O_{1,m,\delta}^<(r;W)\big|_{\nu\to-\nu}\,, \\
&O_{2,0,\delta}^<(r;W) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{-\delta/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{2\delta}, \beta_{2\delta}; \gamma_{2\delta}; x) = \\
&= O_{1,0,-\delta}^<(r;W) = O_{2,0,\delta}^<(r;W)\big|_{\sigma\to-\sigma} = O_{2,0,\delta}^<(r;W)\big|_{\nu\to-\nu}\,,
\end{aligned}$$

$$\begin{aligned}
&O_{3,0,\delta}^<(r;W) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\delta/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_3; 1-x) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\delta/2}(1-x)^{1/4+\nu}\times \\
&\times\left[\frac{\Gamma(\gamma_3)\Gamma(-\delta)}{\Gamma(\alpha_{2\delta})\Gamma(\beta_{2\delta})}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_{1\delta}; x) + x^{-\delta}\frac{\Gamma(\gamma_3)\Gamma(\delta)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}\mathcal{F}(\alpha_{2\delta}, \beta_{2\delta}; \gamma_{2\delta}; x)\right] = \\
&= \frac{\Gamma(\gamma_3)\Gamma(\gamma_{1\delta})}{\Gamma(\alpha_{2\delta})\Gamma(\beta_{2\delta})}O_{1,0,\delta}^<(r;W) + \frac{\Gamma(\gamma_3)\Gamma(\delta)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}O_{2,0,\delta}^<(r;W), \\
&v_{0,\delta}^<(r;W) = \frac{R^2\sqrt{r}}{R^2 - r^2}x^{\delta/2}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_{4\delta}, \beta_{4\delta}; \gamma_4; 1-x) = v_{0,\delta}(r;W)\big|_{\nu\to-\nu}\,,
\end{aligned}$$

$$\begin{aligned}
&\alpha_{1\delta,2\delta} = 1/2 \pm \delta/2 + \nu + \sigma,\ \beta_{1\delta,2\delta} = 1/2 \pm \delta/2 + \nu - \sigma, \\
&\alpha_{4\delta} = 1/2 + \delta/2 - \nu + \sigma,\ \beta_{4\delta} = 1/2 + \delta/2 - \nu - \sigma, \\
&\gamma_{1\delta,2\delta} = 1 \pm \delta,\ \gamma_{3,4} = 1 \pm 2\nu.
\end{aligned}$$

Note that $O^<_{1,0,\delta}(r;W)$ and $O^<_{2,0,\delta}(r;W)$ are real-entire in W solutions of eq. (2.25).

Represent $O^<_{3,0,\delta}$ in the form

$$
\begin{aligned}
&O^<_{3,0,\delta}(r;W) = B_{0,\delta}(W)O^<_{1,0,\delta}(r;W) - \frac{\Gamma(\gamma_3)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}O^<_{4,0,\delta}(r;W),\\
&O^<_{4,0,\delta}(r;W) = \Gamma(\delta)\left[O^<_{1,0,\delta}(r;W) - O^<_{1,0,-\delta}(r;W)\right],\\
&B_{0,\delta}(W) = \frac{\Gamma(\gamma_3)\Gamma(-\delta)}{\Gamma(\alpha_{2\delta})\Gamma(\beta_{2\delta})} + \frac{\Gamma(\gamma_3)\Gamma(\delta)}{\Gamma(\alpha_{1\delta})\Gamma(\beta_{1\delta})}.
\end{aligned}
$$

We obtain the solution of eq.(2.24) as the limit $\delta \to 0$ of the solution of (2.25):

$$
\begin{aligned}
&O^<_{1,0}(r;W) = O^<_{1,0,0}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1,\beta_1;1;x) =\\
&(\text{for } \operatorname{Im} W > 0) = \frac{\Gamma(-2\nu)}{\Gamma(\alpha_4)\Gamma(\beta_4)}O^<_{3,0}(r;W) + \frac{\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}v^<_0(r;W),\\
&v^<_0(r;W) = v^<_{0,0}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_4,\beta_4;\gamma_4;1-x),\\
&O^<_{4,0}(r;W) = \lim_{\delta\to 0} O^<_{4,0,\delta}(r;W) = 2\lim_{\delta\to 0}\partial_\delta O^<_{1,0,\delta}(r;W),\\
&O^<_{3,0}(r;W) = \frac{R^2\sqrt{r}}{R^2-r^2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1,\beta_1;\gamma_3;1-x) =\\
&= B_0(W)O_{1,0}(r;W) - C_0(W)O_{4,0}(r;W),\ C_0(W) = \frac{\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)},\\
&B_0(W) = B_{0,0}(W) = C_0(W)f_0(W),\ f_0(W) = \left[2\psi(1) - \psi(\alpha_1) - \psi(\beta_1)\right].
\end{aligned}
$$

Note that $O^<_{1,0}(r;W)$ and $O^<_{4,0}(r;W)$ are real-entire in W.

2.4.2 Asymptotics, $r \to 0$ ($x \to 0$)

We have

$$
\begin{aligned}
&O^<_{1,0}(r;W) = u_{1\text{as}}(r)(1+O(r^2)),\ O^<_{4,0}(r;W) = u_{4\text{as}}(r)\left(1+O(r^2)\right),\\
&u_{1\text{as}}(r) = r^{1/2},\ u_{4\text{as}}(r) = r^{1/2}\ln\left(\frac{4r^2}{R^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
&O^<_{3,0}(r;W) = .C_0(W)\left[f_0(W)u_{1\text{as}}(r) - C_0(W)u_{4\text{as}}(r)\right]\left(1+O(r^2)\right),\\
&\operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

2.4.3 Asymptotics, $\Delta = R - r \to 0$ ($\delta = 1 - x \to 0$)

We have

$$
O^<_{3,0}(r;W) = \frac{1}{2}R^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),
$$

$$
\begin{aligned}
&O^<_{1,0}(r;W) = \frac{\Gamma(2\nu)}{2\Gamma(\alpha_1)\Gamma(\beta_1)}R^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)),\\
&\operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

2.4.4 Wronskian

We have

$$
\begin{aligned}
&\mathrm{Wr}(O^{<}_{1,0}, O^{<}_{4,0}) = 2,\\
&\mathrm{Wr}(O^{<}_{1,0}, O^{<}_{4,0}) = -\frac{2\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)} = -2C_0(W).
\end{aligned}
$$

Note that any solutions of eq. (2.24) are s.-integrable in the origin and only one solution ($O^{<}_{3,0}$) is s.-integrable on the right end R (for $\operatorname{Im} W > 0$), such that there is one solution ($O^{<}_{3,0}$) belonging to $L^2(\mathbb{R}_+)$ for $\operatorname{Im} W > 0$ and the deficiency indexes of the symmetric operator $\check{h}_0$ (see below) are equal to $(1, 1)$.

2.4.5 Symmetric operator $\hat{h}_{0,\zeta}$

For given a differential operation $\check{h}_0$, the symmetric operator $\hat{h}_{0,\zeta}$ is given by eq. 2.7 with substitution 0 for m.

2.4.6 Isometry

All considerations of subsec 2.4 retain hold with substitution 0 for m. Further, we consider the interval $(0, R)$.

2.4.7 Adjoint operator $\hat{h}_0^+ = \hat{h}_0^*$

We will omit the superscript (1/2).and will write r for ξ, $0 \le r \le R$.

It is easy to prove by standard way that the adjoint operator $\hat{h}_0^+$ coincides with the operator $\hat{h}_0^*$,

$$
\hat{h}_0^+ : \left\{ \begin{array}{l} D_{h_0^+} = D^*_{\check{h}_0}(0,R) = \{\psi_*, \psi_*' \text{ are a.c. in } (0,R),\ \psi_*, \hat{h}_0^+\psi_* \in L^2(0,R)\} \\ \hat{h}_0^+\psi_*(r) = \check{h}_0\psi_*(r),\ \forall \psi_* \in D_{h_0^+} \end{array} \right. .
$$

2.4.8 Asymptotics

Because $\check{h}_0\psi_* \in L^2(0, R)$, we have

$$
\check{h}_0\psi_*(r) = \eta(r),\ \eta \in L^2(0, R),
$$

and we can represent ψ_* in the form

$$
\begin{aligned}
&\psi_*(r) = c_1 O^{<}_{1,0}(r;0) + c_2 O^{<}_{3,0}(r;0) + I(r),\\
&\psi_*'(r) = c_1 \partial_r O^{<}_{1,0}(r;0) + c_2 \partial_r O^{<}_{3,0}(r;0) + I'(r),
\end{aligned}
$$

where

$$
\begin{aligned}
I(r) &= \frac{O^{<}_{3,0}(r;0)}{2C_0(0)} \int_0^r O^{<}_{1,0}(y;0)\eta(y)dy - \frac{O^{<}_{1,0}(r;0)}{2C_0(0)} \int_0^r O^{<}_{3,0}(y;0)\eta(y)dy,\\
I'(r) &= \frac{\partial_r O^{<}_{3,0}(r;0)}{2C_0(0)} \int_0^r O^{<}_{1,0}(y;0)\eta(y)dy - \frac{\partial_r O^{<}_{1,0}(r;0)}{2C_0(0)} \int_0^r O^{<}_{3,0}(y;0)\eta(y)dy.
\end{aligned}
$$

I) $r \to 0$

We obtain with the help of the CB-inequality:

$$I(r) = O(r^{3/2}\ln r),\ I'(r) = O(r^{1/2}\ln r),$$

such that we have

$$\begin{aligned}\psi_*(r) &= c_1 u_{1\mathrm{as}}(r) + c_2 u_{4\mathrm{as}}(r) + O(r^{3/2}\ln r),\\ \psi_*'(r) &= c_1 u_{1\mathrm{as}}'(r) + c_2 u_{4\mathrm{as}}'(r) + O(r^{1/2}\ln r).\end{aligned}$$

II) $r \to R$

In this case, we prove that $[\psi_*, \chi_*]^R = 0$, $\forall \psi_*, \chi_* \in D_{h_m^+}$.

2.4.9 Self-adjoint hamiltonians $\hat{h}_{0\theta}$

The calculation of the asymetry form $\Delta_{h_0^+}(\psi_*)$ gives

$$\begin{aligned}&\Delta_{h_m^+}(\psi_*) = -[\psi_*, \psi_*]_0 = p_0(r)\left[\overline{\psi_*(r)}\psi_*'(r) - \overline{\psi_*'(r)}\psi_*(r)\right]\Big|_{r\to 0} =\\ &= 2(\overline{c_1}c_2 - \overline{c_2}c_1) = -i(.\overline{c_+}c_+ - \overline{c_-}c_-),\ c_\pm = c_1 \pm ic_2.\end{aligned}$$

The condition $\Delta_{h_0^+}(\psi) = 0$ gives

$$\begin{aligned}&c_- = -e^{2i\theta}c_+,\ |\theta| \le \pi/2,\ \theta = -\pi/2 \sim \theta = \pi/2,\ \Longrightarrow\\ &\Longrightarrow\ c_1\cos\theta = c_2\sin\theta,\end{aligned}$$

or

$$\begin{aligned}&\psi(r) = C\psi_{\theta\mathrm{as}}(r) + O(r^{3/2}\ln r),\ \psi'(r) = C\psi_{\theta\mathrm{as}}'(r) + O(r^{1/2}\ln r),\\ &\psi_{\theta\mathrm{as}}(r) = u_{1\mathrm{as}}(r)\sin\theta + u_{4\mathrm{as}}(r)\cos\theta\end{aligned}\tag{2.27}$$

(in this section we write θ instead of more right but more cumbersome θ_ζ). We thus have a family of s.a. hamiltohians $\hat{h}_{0\theta}$,

$$\hat{h}_{0\theta}:\begin{cases} D_{h_{0\theta}} = \{\psi \in D_{h_0^+},\ \psi \text{ satisfy the boundary condition (2.27)}\\ \hat{h}_{0\theta}\psi = \check{h}_0\psi,\ \forall\psi \in D_{h_{0\theta}}\end{cases}.\tag{2.28}$$

2.4.10 The guiding functional

As a guiding functional $\Phi_\theta(\xi; W)$ we choose

$$\begin{aligned}&\Phi_{0\theta}(\xi; W) = \int_0^R U_{0\theta}(r; W)\xi(r)dr,\ \xi \in \mathbb{D}_\theta = D_r(0, R) \cap D_{h_{0\theta}}.\\ &.U_{0\theta}(r; W) = O_{1,0}^<(r; W)\sin\theta + O_{4,0}^<(r; W)\cos\theta,\end{aligned}\tag{2.29}$$

$U_{0\theta}(r; W)$ is real-entire solution of eq. (2.24) satisfying the boundary condition (2.27).

The guiding functional $\Phi_{0\theta}(\xi; W)$ is simple and the spectrum of $\hat{h}_{0\theta}$ is simple.

2.4.11 Green function $G_{0\theta}(r,y;W)$, spectral function $\sigma_{0\theta}(E)$

We find the Green function $G_{0\theta}(r,y;W)$ as the kernel of the integral representation

$$\psi(r)=\int_0^\infty G_{0\theta}(r,y;W)\eta(y)dy,\ \eta\in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{0\theta}-W)\psi(r)=\eta(r),\ \operatorname{Im}W>0, \tag{2.30}$$

for $\psi\in D_{h_{0\theta}}$. General solution of eq. (2.30) (under condition $\psi\in L^2(0,R)$) can be represented in the form

$$\begin{aligned}
&\psi(r)=aO^<_{3,0}(r;W)+\frac{O^<_{1,0}(r;W)}{2C_0(W)}\eta_3(W)+I(r),\ \eta_3(W)=\int_0^R O^<_{3,0}(y;W)\eta(y)dy,\\
&I(r)=\frac{O^<_{3,0}(r;W)}{2C_0(W)}\int_0^r O^<_{1,0}(y;W)\eta(y)dy-\frac{O^<_{1,0}(r;W)}{2C_0(W)}\int_0^r O^<_{3,0}(y;W)\eta(y)dy,\\
&I(r)=O\left(r^{3/2}\ln r\right),\ r\to 0.
\end{aligned}$$

A condition $\psi\in D_{h_{0\theta}}$, (i.e.ψ satisfies the boundary condition (2.27)) gives

$$a=-\frac{\cos\theta}{2C_0^2(W)\omega_\theta(W)}\eta_3(W),\ \omega_\theta(W)=f_0(W)\cos\zeta+\sin\zeta,$$

$$\begin{aligned}
&G_{0\theta}(r,y;W)=\pi\Omega_{0\theta}(W)U_{0\theta}(r;W)U_{0\theta}(y;W)+\\
&+\frac{1}{2}\begin{cases}\tilde{U}_{0\theta}(r;W)U_{0\theta}(y;W),\ r>y\\ U_{0\theta}(r;W)\tilde{U}_{0\theta}(y;W),\ r<y\end{cases},\\
&\Omega_{0\theta}(W)=\frac{\tilde{\omega}_\theta(W)}{2\pi\omega_\theta(W)},\ \tilde{\omega}_\theta(W)=f_0(W)\sin\theta-\cos\zeta,\\
&U_{0\theta}(y;W)=O^<_{1,0}(r;W)\sin\theta+O^<_{4,0}(r;W)\cos\theta,\\
&\tilde{U}_{0\theta}(r;W)=O^<_{1,0}(r;W)\cos\theta-O^<_{4,0}(r;W)\sin\theta,
\end{aligned} \tag{2.31}$$

where we used an equality

$$O^<_{3,0}(r;W)=C_0(W)[\tilde{\omega}_\theta(W)U_{0\theta}(r;W)+\omega_\theta(W)\tilde{U}_{0\theta}(r;W)].$$

Note that the functions $U_{0\theta}(r;W)$ and $\tilde{U}_{0\theta}(r;W)$ are real-entire in W and the last term in the r.h.s. of eq. (2.31) is real for $W=E$. For $\sigma'_{0\theta}(E)$, we find

$$\sigma'_{0\theta}(E)=\operatorname{Im}\Omega_{0\theta}(E+i0). \tag{2.32}$$

2.4.12 Spectrum

2.4.13 $\theta=\pi/2$

First we consider the case $\theta=\pi/2$.

In this case, we have $U_{0\pi/2}(r;W)=O^<_{1,0}(r;W)$ and

$$\sigma'_{0\pi/2}(E)=\frac{1}{2\pi}\operatorname{Im}f(W)|_{W=E+i0}\,.$$

$w = R^2E > q$ In this case, we have $\alpha_1 = 1/2 + \sigma - i\sqrt{w-q}$, $\beta_1 = 1/2 - \sigma - i\sqrt{w-q}$, and it is easy to prove that $\alpha_1, \beta_1 \notin \mathbb{Z}_-$, such that $f(E)$ is finite complex function of E, $f(E) = \mathcal{U}(E) + i\mathcal{V}(E)$, $\mathcal{U}(E) = \operatorname{Re} f(E)$, $\mathcal{V}(E) = \operatorname{Im} f(E) > 0$, and we find

$$\sigma'_{0\pi/2}(E) = \frac{1}{2\pi}\mathcal{V}(E) \equiv \rho^2_{0\pi/2}(E) > 0$$
$$\mathrm{spec}\hat{h}_{0\pi/2} = [q/R^2, \infty),$$

where $f_0(E) = \mathcal{U}_0(E) + i\mathcal{V}_0(E)$, $\mathcal{U}_0(E) = \operatorname{Re} f_0(E)$, $\mathcal{V}_0(E) = \operatorname{Im} f_0(E) > 0$. Note that $\lim_{\Delta\to 0} \varrho_{0\pi/2}(E) = 0$ for $q \neq q_{0,k}$ and $\varrho_{0\pi/2}(E) = O(\Delta^{1/4})$ for $q = q_{0,k}$, $\Delta = E - q/R^2 \to +0$, $q_{0,k} = 4N^2_{0,k}$, $N_{0,k} = 1 + 2k$

$w = R^2E = q + \Delta$, $\Delta \sim 0$ In this case, we have $\nu = \sqrt{-\Delta}$,

$q \geq 0$ In this case, we have

$$\operatorname{Im}\psi(\alpha_1) = \begin{cases} -O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases}.$$

i) $q \neq q_{0,k}$

$$\operatorname{Im}\psi(\beta_1) = \begin{cases} -O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases} \Longrightarrow$$

$$\sigma'_{0\pi/2}(E) = \begin{cases} O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases}.$$

ii) $q = q_{0,k}$, $\beta_1 = -k + \sqrt{-\Delta}/4$, $k = 0, 1, ...$

$$\operatorname{Im}\psi(\beta_1) = \begin{cases} -\frac{1}{2R\sqrt{\Delta}} + O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases} \Longrightarrow$$

$$\sigma'_{0\pi/2}(E) = \begin{cases} -\frac{1}{2R\sqrt{\Delta}} + O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases}.$$

$q < 0$ In this case, we have $\alpha_1 = 1/2 + \sqrt{-\Delta} + i\varkappa$, $\beta_1 = 1/2 + \sqrt{-\Delta} - i\varkappa$, and

$$\sigma'_{0\pi/2}(E) = \begin{cases} O(\sqrt{\Delta}).\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases}.$$

$w = R^2E < q$ In this case, we have $\operatorname{Im}\nu|_{W=E} = 0$, $.\nu|_{W=E} > 0$.

$q > 0$ In this case, we have $\operatorname{Im}\alpha_1|_{W=E} = \operatorname{Im}\beta_1|_{W=E} = 0$, $\operatorname{Im}\psi(\alpha_1)|_{W=E} = 0$, and

$$\sigma'_{0\pi/2}(E) = -\frac{1}{2\pi}\operatorname{Im}\psi(\beta_1)|_{W=E+i0}.$$

$\sigma'_{0\pi/2}(E)$ can be different from zero in the points $\mathcal{E}_{0n}$ when $\beta_1 = -n$, $n = 0, 1, 2, ...$, i.e., $\mathcal{E}_{0n}$ satisfy the equations

$$\sqrt{q - w_{0\pi/2,n}} = \sqrt{q} - 2N_{0,n},\ N_{0,n} = 1 + 2n \implies \tag{2.33}$$
$$\implies \mathcal{E}_{0n} = \frac{q}{R^2} - \frac{\left(\sqrt{q} - 2N_{0,n}\right)^2}{R^2} = \frac{4\sqrt{q}N_{0,n}}{R^2} - \frac{4N_{0,n}^2}{R^2}.$$

Eq. (2.33) has solutions only if $q \geq q_{0,0}$, $q_{0,0} = 4N_{0,0}^2 = q$. Then we have: $n = 0, 1, ..., n_{\max}$, $\mathcal{E}_{0n} > \mathcal{E}_{0n-1}$, $n = 1, ..., n_{\max}$, $0 < \mathcal{E}_{0n} \leq q/R^2$, where

$$\sigma'_{0\pi/2}(E) = \sum_{n=0}^{n_{\max}} Q_{0\pi/2,n}^2 \delta(E - \mathcal{E}_n),\ Q_{0\pi/2,n} = 2R^{-1}\left(q - w_{0\pi/2,n}\right)^{1/4} ..$$

$$\left\{ \begin{array}{l} q \leq q_{0,0},\ \text{no levels} \\ n_{\max} = k,\ \sqrt{q} = 4[1/2 + k + \delta) \end{array} \right. ,\ k = 0.1, ...,\ 0 < \delta \leq 1,$$

The level $\mathcal{E}_{0n_{\max}+1} = \mathcal{E}_{0k+1}$ would be equal to q/R^2 for $q = q_{0,k+1} = 4N_{0,k+1}^2$. But, in this case, we have $Q_{0\pi/2,k+1} = 0$, such that the level $\mathcal{E}_{0n_{\max}+1} = \mathcal{E}_{0k+1}$ is factually absent. Analogously, we see that the level $\mathcal{E}_{00}$ for $q = q_{0,0}$ is factually absent. Thus, the discrete part of the spectrum of $\hat{h}_{0\pi/2}$ is simple and has the form

$$\text{spec}\hat{h}_{0\pi/2} = \{\mathcal{E}_{0n},\ 0 < \mathcal{E}_{0n} < q/R^2,\ n = 0, 1, ..., n_{\max}\},$$
$$n_{\max} = k \text{ for } \sqrt{q} = 4[\frac{1}{2} + k + \delta],\ 0 < \delta \leq 1,\ k \in \mathbb{Z}_+$$

The discrete part of the spectrum is absent for $q \leq q_{0,0}$.

$q \leq 0$ In this case, we have $\alpha_1 = 1/2 + \nu + i\varkappa$, $.\beta_1 = 1/2 + \nu - i\varkappa$, and

$$\sigma'_{0\pi/2}(E) = 0.$$

Finally, we find.

The spectrum of $\hat{h}_{0\pi/2}$ is simple, $\text{spec}\hat{h}_{0\pi/2} = [q/R^2, \infty) \cup \{\mathcal{E}_{0n},\ n = 0, 1, ...n_{\max}\}$, the discrete part of spectrum is present for $q > q_{0,0}$. The set of functions

$$\left\{U_{0\pi/2}(r; E) = \varrho_{0\pi/2}(E) O_{10}^{<}(r; E),\ E \geq q/R^2;\ U_{0\pi/2,n}(r) = Q_{0\pi/2,n} O_{10}^{<}(r; \mathcal{E}_{0n}),\ n = 0, 1, ...n_{\max}\right\}$$

forms a complete orthogonalized system in $L^2(\mathbb{R}_+)$.

Note that these results for spectrum and the set of eigenfunctions can be obtained from the corresponding results of sec. 3 by formal substitution $|m| \to 0$.

The same results we obtain for the case $\zeta = -\pi/2$.

2.4.14 $|\theta| < \pi/2$

Now we consider the case $|\theta| < \pi/2$.

In this case, we can represent $\sigma'_\theta(E)$ in the form

$$\sigma'_{0\theta}(E) = -\frac{1}{2\pi\cos^2\theta} \operatorname{Im} \frac{1}{f_\theta(E + i0)},\ f_{0\theta}(W) = f_0(W) + \tan\theta.$$

$w > q$ In this case, we have

$$\sigma'_{0\theta}(E) = \frac{1}{2\pi}\frac{\mathcal{V}_0(E)}{[\mathcal{U}_0(E)\cos\theta + \sin\theta]^2 + \mathcal{V}_0^2(E)\cos^2\theta} \equiv \rho_{0\theta}^2(E).$$

The spectrum of $\hat{h}_{0\theta}$ is simple and continuous, $\mathrm{spec}\hat{h}_{0\theta} = [q/R^2, \infty)$.

$w = q + \tilde{\Delta}$, $\tilde{\Delta} = \Delta + i\varepsilon \sim 0$, $\operatorname{Im}\Delta = 0$ In this case, we have $\alpha_1 = 1/2 + \sigma + \sqrt{-\Delta}/4$, $\beta_1 = 1/2 - \sigma + \sqrt{-\Delta}/4$,

$$\sqrt{-\Delta} = \begin{cases} -i\sqrt{\Delta},\ \Delta \geq 0 \\ \sqrt{|\Delta|},\ \Delta \leq 0 \end{cases},\ \sigma = \begin{cases} \sqrt{q}/4,\ q \geq 0 \\ i\sqrt{|q|}/4,\ q \leq 0 \end{cases}.$$

A direct estimation gives

$$\sigma'_{0\theta}(E) = \begin{cases} \begin{cases} O(\sqrt{\Delta}),\ q = q_{0k} \text{ or } q \neq q_{0k}, \theta \neq \theta_0 \\ O(1/\sqrt{\Delta}),\ \theta = \theta_0, q \neq q_{0k} \end{cases},\ \Delta > 0 \\ 0,\ \Delta < 0 \end{cases},$$

where $\tan\theta_0 = \psi(\alpha_{10}) + \psi(\beta_{10}) - 2\psi(1)$, $\alpha_{10} = 1/2 + \sigma$, $\beta_{10} = 1/2 - \sigma$

$w < q$ In this case, we have $\operatorname{Im}\nu|_{W=E} = 0$, $\nu|_{W=E} > 0$,

$$\alpha_1 = 1/2 + \nu + \sqrt{q}/4,\ \beta_1 = 1/2 - \nu + \sqrt{q}/4,\ q \geq 0,$$
$$\alpha_1 = 1/2 + \nu + i\sqrt{|q|}/4,\ \beta_1 = \overline{\alpha_1},\ q \leq 0.$$

Thus, we have: the function $[f_{0\theta}(E)]^{-1}$ is real except the points $E_{0n}(\theta)$,

$$f_{0\theta}(E_{0n}(\theta)) = 0, \tag{2.34}$$

such that we obtain

$$\sigma'_{0\theta}(E) = \sum_{n\in\mathcal{N}} Q_{0\theta,n}^2\delta(E - E_{0n}(\theta)),\ Q_{0\theta,n} = \frac{1}{\sqrt{2\partial_E f_{0\theta}(E_{0n}(\theta))\cos\theta}},$$
$$\partial_E f(E) = \frac{R^2[\psi'(\alpha_1) + \psi'(\beta_1)]}{8\sqrt{q-w}} > 0, \tag{2.35}$$

where $\mathcal{N}$ is a subset of $\mathbb{Z}$ to be described below. Furthermore, we find

$$\partial_\theta E_{0n}(\theta) = -[\partial_E f_{0\theta}(E_{0n}(\theta))\cos^2\theta]^{-1} < 0.$$

$q_{0,k} < q \leq q_{0,k+1}$, $\sqrt{q}/4 = 1/2 + k + \delta$, $0 < \delta \leq 1$, $k \in \mathbb{Z}_+$ In this case, the function $f_0(E)$ has the properties: $f_0(E) \to -\infty$ as $E \to -\infty$; $f_0(\mathcal{E}_{0n} \pm 0) = \mp\infty$ $n = 0, 1, ..., n_{\max} = k$; $f_0(E) \to -\tan\theta_0 = 2\psi(1) - \psi(1 + k + \delta) - \psi(-k - \delta) - 0$ as $E \to q/R^2 - 0$. We find: in each interval $(\mathcal{E}_{0n-1}, \mathcal{E}_{0n})$, $n = 0, ..., k$ (we set $\mathcal{E}_{0-1} = -\infty$), for any fixed $\theta \in (-\pi/2, \pi/2)$, there is one level $E_{0n}(\theta)$ which run monotonocally from $\mathcal{E}_{0n-1} + 0$ to $\mathcal{E}_{0n} - 0$ as θ run from $\pi/2 - 0$ to -$\pi/2 + 0$; in the interval $(\mathcal{E}_{0k}, q/R^2)$, for any fixed $\theta \in (\theta_0, \pi/2)$, there is one level $E_{0k+1}(\theta)$ which run monotonocally from $\mathcal{E}_{0k} + 0$ to $q/R^2 - 0$ as θ run from $\pi/2 - 0$ to $\theta_0 + 0$; eq. (2.34) has no solutions in the interval $(\mathcal{E}_{0k}, q/R^2)$ for $\theta \in (-\pi/2, \theta_0)$. Formally, eq. (2.34) has solution $E_{0k+1}(\theta_0) = q/R^2$ for $\theta = \theta_0$. However, in this case $Q_{0\theta_0,k} = 0$, as it is follows from eq. (2.35).We find also

$$\mathcal{N} = \mathcal{N}_{0k}(\theta) = \begin{cases} \{0, 1, ..., k\},\ \theta \in (-\pi/2, \theta_0] \\ \{0, 1, ..., k+1\},\ \theta \in (\theta_0, \pi/2) \end{cases}.$$

$\boldsymbol{.} q \leq q_{0,0} = 4$ In this case, the function $f_0(E)$ has the properties: $f_0(E) \to -\infty$ as $E \to -\infty$; $f_0(E) \to -\tan\theta_0 - 0 = 2\psi(1) - \psi(\alpha_1) - \psi(\beta_1) - 0$ as $E \to q/R^2 - 0$;.$f(E)$ increases monotonocally on the interval $(-\infty, q/R^2)$. Then we find: in the interval $(-\infty, q/R^2]$, for any fixed $\theta \in (\theta_0, \pi/2)$, there is one level $E_{0-1}(\theta)$ which run monotonocally from $-\infty$ to $q/R^2 - 0$ as θ run from $\pi/2 - 0$ to $\theta_0 + 0$; there are no discrete levels on the interval $(-\infty, q/R^2]$ for $\theta \in (-\pi/2, \theta_0]$. We find also

$$\mathcal{N} = \mathcal{N}_{0-1}(\theta) = \begin{cases} \varnothing, \ \theta \in (-\pi/2, \theta_0] \\ \{-1\}, \ \theta \in (\theta_0, \pi/2) \end{cases} .$$

Finally, we obtain. The spectrum of $\hat{h}_{0\theta}$ is simple and spec$\hat{h}_{0\theta} = [q/R^2, \infty) \cup \{E_{0n}(\theta), \ n \in \mathcal{N}\}$. The set of functions

$$\left\{U_{0\theta}(r; E) = \varrho_{0\theta}(E) O_{10}^{<}(r; E), \ E \geq q/R^2; \ U_{0\theta,n}(r) = Q_{0\theta,n} U_\theta(r; E_n(\theta),), \ n \in \mathcal{N}\right\}$$

forms a complete orthogonalized system in $L^2(0, R)$.

2.5 S.a. operators $\hat{H}_{\mathfrak{e}m,\zeta}$

Then s.a. operator $\hat{H}_{\mathfrak{e}m,\zeta}$can be reconstructed by the rules of subsec. 4 Isometry of sec. 3 from operators $\hat{h}_{m,\zeta}^{(1/2)}$ which are equal to the operators $\hat{h}_{\mathfrak{e}m}$ of sec. 3 for $|m| \geq 1$ and to the operators $\hat{h}_{0\theta}$ of sec. 4 for $m = 0$. We do not write explicitly the corresponding formulas and note only that the parameter θ of s.a. hamiltonians $\hat{h}_{0\theta}$ should be dependent of ζ, $\theta = \theta(\zeta)$. Note also that a QM-systems obtained can be considered (for fixed $\mathfrak{H}_{m,\zeta}$ and θ_ζ) as union of two isomorphic noninteracting systems placed on two sheets of the huperboloid. The absent of interaction is due to the vanishind of the current density through the boundary (through the points $r = 0, R - 0, R + 0, r = \infty$).

3 Quantum two-dimensional Coulomb-like interaction on pseudosphere

Consider a space with coordinates ρ, $0 < \rho < \infty$, and ϕ, $0 \leq \phi \leq 4\pi$, $\phi = 0 \sim \phi = 4\pi$. Let a differential operation $\check{H}_C = \check{H}$,

$$\begin{aligned} &\check{H} = -\Delta_{BL\rho} - \Delta_{BL\phi} + V_C, \ \Delta_{BL\rho} = \frac{(R_C^2 - \rho^2)^2}{R_C^4} \frac{1}{\rho} \partial_\rho \rho \partial_\rho, \\ &\Delta_{BL\phi} = \frac{(R_C^2 - \rho^2)^2}{R_C^4} \frac{1}{\rho^2} \partial_\phi^2, \ V_C = \frac{g(R_C + \rho)^2}{R_C^2 \rho}, \end{aligned} \tag{3.1}$$

be given. Consider the equation

$$\begin{aligned} &(\check{H} - W_C)\Psi(\rho, \phi) = 0, \ \rho < R_C, \\ &\Psi(\rho, \phi) = \sum_{m \in \mathbb{Z}} \Lambda_m(\rho, \phi), \ \Lambda_m(\rho, \phi) = \frac{1}{\sqrt{4\pi}} e^{im\phi/2} \Psi_m(\rho) \implies \\ &\left(\check{H}_{Cm} - W_C\right) \Psi_m(\rho) = 0, \ \check{H}_{Cm} = \check{H}_m = -\Delta_{BL\rho} + \frac{m^2 (R_C^2 - \rho^2)^2}{4 R_C^4 \rho^2} + V_C \end{aligned} \tag{3.2}$$

Represent $\Psi_m(\rho)$ in the form

$$\Psi_m(\rho) = x^{\tilde{\mu}}(1-x)^{1/4+\tilde{\nu}}\Phi_{m,\xi_\mu,\xi_\nu}(x),\ \tilde{\mu} = \xi_\mu\mu,\ \tilde{\nu} = \xi_\nu\nu,\ \xi_\mu, \xi_\nu = \pm 1,$$

where

$$x_C = x = \frac{4R_C\rho}{(R_C+\rho)^2},\ \rho = R_C\frac{(1-\sqrt{1-x})^2}{x},\ \frac{\partial\rho}{\partial x} = \frac{\rho}{x\sqrt{1-x}},$$
$$\rho\partial_\rho = x\sqrt{1-x}\partial_x,\ \frac{(R_C^2-\rho^2)^2}{R_C^4\rho^2} = \frac{16(1-x)}{R_C^2x^2},\ V_C = \frac{16(1-x)}{R_C^2x}\frac{gR_C}{4(1-x)},$$
$$\Delta_{BL\rho} = \frac{16(1-x)^{3/2}}{R_C^2x}\partial_x x\sqrt{1-x}\partial_x = \frac{1}{x}\Delta_{BLr},\ \Delta_{BL\phi} = \frac{16(1-x)}{R_C^2x}\left(-\frac{(m+\delta)^2}{4x}\right).$$

Then we have

$$\left[x(1-x)\partial_x^2 + (\gamma_{\xi_\mu} - (1+\alpha_{\xi_\mu,\xi_\nu}+\beta_{\xi_\mu,\xi_\nu})x)\partial_x - \alpha_{\xi_\mu,\xi_\nu}\beta_{\xi_\mu,\xi_\nu}\right]\Phi_{m,\xi_\mu,\xi_\nu} = 0,$$
$$\alpha_{\xi_\mu,\xi_\nu} = 1/2+\tilde{\mu}+\tilde{\nu}+\sigma,\ \beta_{\xi_\mu,\xi_\nu} = 1/2+\tilde{\mu}+\tilde{\nu}-\sigma,\ \gamma_{\xi_\mu} = 1+2\tilde{\mu},$$
$$\mu = \frac{|m|}{2},\ \nu = \frac{1}{4}\sqrt{1+4R_Cg-w_C},\ \sigma = \frac{1}{4}\sqrt{1-w_C},$$
$$w_C = R_C^2W_C.$$

It is convenient to solve first more general equation

$$\left(\check{H}_{Cm,\delta} - W_C\right)\Psi_{Cm,\delta}(\rho) = 0. \check{H}_{Cm,\delta} = \check{H}_{m,\delta} = -\Delta_{BL\rho} + \frac{(m+\delta)^2(R_C^2-\rho^2)^2}{4R_C^4\rho^2} + V_C \quad (3.3)$$

Represent $\Psi_{Cm,\delta}(\rho) = \Psi_{m,\delta}(\rho)$ in the form

$$\Psi_{m,\delta}(\rho) = x^{\tilde{\mu}_\delta}(1-x)^{1/4+\tilde{\nu}}\Phi_{m,\xi_\mu,\xi_\nu,\delta}(x),\ \tilde{\mu} = \xi_\mu\mu,\ \tilde{\nu} = \xi_\nu\nu,\ \xi_\mu, \xi_\nu = \pm 1,$$

where

$$\tilde{\mu}_\delta = \xi_\mu\mu_\delta,\ \mu_\delta = \frac{|m+\delta|}{2}.$$

Then we have

$$\left[x(1-x)\partial_x^2 + (\gamma_{\xi_\mu,\delta} - (1+\alpha_{\xi_\mu,\xi_\nu,\delta}+\beta_{\xi_\mu,\xi_\nu,\delta})x)\partial_x - \alpha_{\xi_\mu,\xi_\nu,\delta}\beta_{\xi_\mu,\xi_\nu,\delta}\right]\Phi_{m,\xi_\mu,\xi_\nu,\delta} = 0,$$
$$\alpha_{\xi_\mu,\xi_\nu,\delta} = 1/2+\tilde{\mu}_\delta\tilde{\nu}+\sigma,\ \beta_{\xi_\mu,\xi_\nu,\delta} = 1/2+\tilde{\mu}_\delta+\tilde{\nu}-\sigma,\ \gamma_{\xi_\mu,\delta} = 1+2\tilde{\mu}_\delta.$$

3.1 Scalar product

Differential operation (3.1) is s.a. for the standard scalar product on the pseudosphere

$$d\Lambda(\rho,\phi) = d\omega(\rho)d\phi,\ d\omega(r) = \frac{R_C^4\rho}{(R_C^2-\rho^2)^2}d\rho,$$

$$(\Lambda_{1m_1}, \Lambda_{2m_2}) = \delta_{m_1m_2}\left(\Lambda_{1m_1}\Lambda_{2m_1}\right),\ (\Lambda_{1m}, \Lambda_{2m}) =$$
$$= (\Psi_{1m}, \Psi_{2m}) = \int_0^{R_C}\overline{\Psi_{1m}(\rho)}\Psi_{2m}(\rho)d\omega(\rho)$$

Represent $\Psi_m(r)$ in the form

$$\Psi_m(r)=\frac{R_C^2-\rho^2}{R_C^2\sqrt{\rho}}\psi_m(\rho).$$

Then we have

$$\begin{aligned}
&(\Lambda_{1m},\Lambda_{2m})=\langle\psi_{1m},\psi_{2m}\rangle=\int_0^{R_C}\overline{\psi_{1m}(\rho)}\psi_{2m}(\rho)d\rho,\\
&\check{H}_m\Psi_m(r)=\frac{R_C^2-\rho^2}{R_C^2\sqrt{\rho}}\check{h}_m\psi_m(\rho),\\
&\check{h}_m=\frac{\sqrt{\rho}}{R_C^2-\rho^2}\check{H}_m\frac{R_C^2-\rho^2}{\sqrt{\rho}}=-\partial_\rho p_0(\rho)\partial_\rho+\\
&+\frac{10R_C^2\rho^2-9\rho^4-R_C^4+m^2(R_C^2-\rho^2)^2}{4R_C^4r^2}+V(\rho),\ p_0(\rho)=\frac{(R_C^2-\rho^2)^2}{R_C^4}.
\end{aligned}\tag{3.4}$$

We introduce also the differential operation $\check{h}_{m,\delta}$,

$$\check{h}_{m,\delta}=\frac{\sqrt{\rho}}{R_C^2-\rho^2}\check{H}_{m,\delta}\frac{R_C^2-\rho^2}{\sqrt{\rho}}=\left.\check{h}_{m,\delta}\right|_{m\to m+\delta},$$

and corresponding the Shroedinger equations

$$(\check{h}_m-W)\psi_m(\rho)=0,\tag{3.5}$$

$$(\check{h}_{m,\delta}-W)\psi_{m,\delta}(\rho)=0.\tag{3.6}$$

3.2 Connections and coincidences

In this section, we describe a connection of the oscillator and Coulomb problems on the two-dimensional pseudospheres.

The duality of these theories was demonstrated in [9].

The coordinates of two pseudospheres are connected by so called Levi-Civita–Bohlini transformation

$$\rho=\kappa_0r^2,\ r=\sqrt{\rho/\kappa_0},\ R_C=\kappa_0R^2,\ \phi=2\varphi,$$

and the parameters of two problems satisfy the relations

$$W_O=-4\kappa_0g,\ g=-\frac{W_O}{4\kappa_0},\ q=1-R_C^2W_C,\ W_C=\frac{1-q}{R_C^2}=\frac{1}{R_C^2}-\frac{\lambda}{4\kappa_0^2}.$$

We have

$$\begin{aligned}
&\Delta_{BLr}=\mathcal{R}\Delta_{BL\rho},\ \mathcal{R}=\frac{4\kappa_0^2R^4r^2}{(R^2+r^2)^2}=\frac{4\kappa_0R_C^2\rho}{(R_C+\rho)^2},\ \Delta_{BL\varphi}=\mathcal{R}\Delta_{BL\phi},\\
&W_O-V_O(r)=\mathcal{R}\left[W_C-V_C(\rho)\right],\ \check{H}_O-W_O=\mathcal{R}\left[\check{H}_C-W_C\right],\\
&\check{H}_{Om}-W_O=\mathcal{R}\left[\check{H}_{Cm}-W_C\right],\ \check{H}_{Om,\delta}-W_O=\mathcal{R}\left[\check{H}_{Cm,\delta}-W_C\right],
\end{aligned}$$

$$4gR_C = -R^2 W_O = -w_O,\ w_C = 1 - q,$$
$$\mu_C = \mu_O,\ \mu_{C,\delta} = \mu_{O,\delta},\ \nu_C = \nu_O,\ \sigma_C = \sigma_O,$$
$$\alpha_{C\xi_\mu,\xi_\nu,\delta} = \alpha_{O,\xi_\mu,\xi_\nu,\delta},\ \beta_{C,\xi_\mu,\xi_\nu,\delta} = \beta_{O,\xi_\mu,\xi_\nu,\delta},\ \gamma_{C\xi_\mu,\delta} = \gamma_{O,\xi_\mu,\delta},$$
$$x_C = \frac{4R_C\rho}{(R_C+\rho)^2} = \frac{4R^2r^2}{(R^2+r^2)^2} = x_O.$$

$$A_{Cm,\delta}(W_C) = A_{Om,\delta}(W_O),\ B_{Cm}(W_C) = B_{Om}(W_O),\ |m| \geq 1,$$
$$C_{Cm}(W_C) = C_{Om}(W_O),\ \omega_{Cm}(W_C) = \omega_{Om}(W_O),\ |m| \geq 1.$$

Connection of functions $C(\rho)$ and $O(r)$:

$$C(\rho) = A(\rho)O(r),\ A(\rho) = \frac{R_C^2\rho^{1/2}}{R_C^2-\rho^2}\frac{R^2-r^2}{R^2r^{1/2}} = \frac{R^2(\kappa_0 r)^{1/2}}{R^2+r^2} = \frac{R_C(\kappa_0\rho)^{1/4}}{R_C+\rho}, \tag{3.7}$$
$$O(r) = B(r)C(\rho),\ B(r) = \frac{R^2r^{1/2}}{R^2-r^2}\frac{R_C^2-\rho^2}{R_C^2\rho^{1/2}} = \frac{R_C+\rho}{R_C(\kappa_0\rho)^{1/4}} = \frac{R^2+r^2}{R^2(\kappa_0 r)^{1/2}},$$

$$\mathrm{Wr}_\rho(C_1,C_2) = \frac{1}{2}\mathrm{Wr}_r(O_1,O_2)$$

Connections of asymptotics of functions $C(\rho)$ and $O(r)$:
i) $r \to 0,\ \rho \to 0$

$$C(\rho) = (\kappa_0 r)^{1/2}O(r),\ O(r) = (\kappa_0\rho)^{-1/4}C(\rho).$$

ii) $R - r = \Delta_O \to 0,\ R_C - \rho = \Delta_C \to 0$

$$\Delta_C = 2\kappa_0 R\Delta_O,\ \Delta_O = (4\kappa_0 R_C)^{-1/2}\Delta_C,$$
$$C(\rho) = \frac{1}{2}(\kappa_0 R)^{1/2}O(r),\ O(r) = 2(\kappa_0 R_C)^{-1/4}C(\rho).$$

One of the results of these considerations is the following.

Let $\Psi_{Om}(r) = x_O^{\tilde{\mu}_O}(1-x_O)^{1/4+\tilde{\nu}_O}\Phi_m(x_O)$ is the solution of equation

$$(\check{H}_{Om} - W_O)\Psi_{Om}(r) = 0.$$

Then the function $\Psi_{Cm}(\rho) = x_C^{\tilde{\mu}_C}(1-x_C)^{1/4+\tilde{\nu}_C}\Phi_m(x_C)$ is the solution of eq. (3.2) and reverse.

Let $\Psi_{Om\delta}(r) = x_O^{\tilde{\mu}_{O\delta}}(1-x_O)^{1/4+\tilde{\nu}_O}\Phi_{m\delta}(x_O)$ is the solution of eq. (2.12). Then the function $\Psi_{Cm\delta}(\rho) = x_C^{\tilde{\mu}_{C\delta}}(1-x_C)^{1/4+\tilde{\nu}_C}\Phi_{m\delta}(x_C)$ is the solution of eq. (3.3) and reverse.

Let $\psi_{Om}(r)$,

$$\psi_{Om}(r) = \frac{R^2\sqrt{r}}{R^2-r^2}\Psi_{Om}(r) = \frac{R^2\sqrt{r}}{R^2-r^2}x_O^{\tilde{\mu}_O}(1-x_O)^{1/4+\tilde{\nu}_O}\Phi_m(x_O), \tag{3.8}$$

is the solution of eq. (2.10). Then the function $\psi_{Cm}(\rho)$,

$$\psi_{Cm}(\rho) = \frac{R_C^2\sqrt{\rho}}{R_C^2-\rho^2}\Psi_{Cm}(\rho) = \frac{R_C^2\sqrt{\rho}}{R_C^2-\rho^2}x_C^{\tilde{\mu}_C}(1-x_C)^{1/4+\tilde{\nu}_C}\Phi_m(x_C), \tag{3.9}$$

is the solution of eq. (3.5) and reverse.

Let $\psi_{Om\delta}(r) = \frac{R^2\sqrt{r}}{R^2-r^2}\Psi_{Om\delta}(r) = \frac{R^2\sqrt{r}}{R^2-r^2}x_O^{\tilde{\mu}_{O\delta}}(1-x_O)^{1/4+\tilde{\nu}_O}\Phi_{m\delta}(x_O)$ is the solution of eq. (2.11). Then the function $\psi_{Cm\delta}(\rho) = \frac{R_C^2\sqrt{\rho}}{R_C^2-\rho^2}\Psi_{Cm\delta}(\rho) = \frac{R_C^2\sqrt{\rho}}{R_C^2-\rho^2}x_C^{\tilde{\mu}_C}(1-x_C)^{1/4+\tilde{\nu}_C}\Phi_{m\delta}(x_C)$ is the solution of eq. (3.6) and reverse.

3.3 $|m| \geq 2$

3.3.1 Useful solutions, $\rho < R_C$

We construct the solutions of eq. (3.5) with the help of correspondence formulas (3.7), (3.8), and (3.9). We have

$$C_{1,m}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{\mu}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; \gamma_1; x) =$$
$$(\text{for } \operatorname{Im} W > 0) = Q_1(W)C_{3,m}(\rho; W) + Q_2(W)v_m(\rho; W),\ \gamma_1 = 1 + 2\mu,$$
$$Q_1(W) = \frac{\Gamma(\gamma_1)\Gamma(-2\nu)}{\Gamma(\alpha_4)\Gamma(\beta_4)},\ Q_2(W) = \frac{\Gamma(\gamma_1)\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)},$$
$$v_m(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{\mu}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_4, \beta_4; \gamma_4; 1-x),$$

$$C_{4,m}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}\frac{R^2 - r^2}{R^2 r^{1/2}} O_{4,m}^{<}(r; W) =$$
$$= \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}(1-x)^{1/4+\nu}\lim_{\delta\to 0}[x^{-\mu_\delta}\mathcal{F}(\alpha_{2\delta}, \beta_{2\delta}; \gamma_{2\delta}; x) -$$
$$- A_{m,\delta}(W)\Gamma(\gamma_{2\delta})x^{\mu_\delta}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_{1\delta}; x)],$$
$$A_{m,\delta}(W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_{1\delta})},$$

$$C_{3,m}(r; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{\mu}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; \gamma_3; 1-x) =$$
$$= B_m(W)C_{1,m}(\rho; W) + C_m(W)C_{4,m}(\rho; W),\ C_m(W) = \frac{\Gamma(\gamma_3)\Gamma(|m|)}{\Gamma(\alpha_1)\Gamma(\beta_1)},$$
$$B_m(W) = \frac{(-1)^{|m|+1}\Gamma(\gamma_3)}{2\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_1)}\left[\psi(\alpha_1) + \psi(\alpha_2) + \psi(\beta_1) + \psi(\beta_2)\right],$$

$$\alpha_{1,2} = 1/2 \pm \mu + \nu + \sigma,\ \beta_{1,2} = 1/2 \pm \mu + \nu - \sigma,$$
$$\alpha_4 = 1/2 + \mu - \nu + \sigma,\ \beta_4 = 1/2 + \mu - \nu - \sigma,$$
$$\gamma_1 = 1 + 2\mu,\ \gamma_{3,4} = 1 \pm 2\nu,\ \mu = |m|/2,\ \sigma = \frac{1}{4}\sqrt{1 - w_C},$$
$$\nu = \frac{1}{4}\sqrt{1 + 4gR_C - w_C},\ w_C = R_C^2 W_C.$$

$C_{1,m}(\rho; W)$ and $C_{4,m}(\rho; W)$ are real-entire in W.

3.3.2 Asymptotics, $\rho \to 0$ $(x \to 0)$

We have

$$x = \frac{4\rho}{R_C}(1 + O(\rho)),\ p_0(\rho) = 1 + O(\rho^2),$$
$$C_{1,m}(\rho; W) = (4/R_C)^{|m|/2}\rho^{1/2+|m|/2}(1 + O(\rho)),$$
$$C_{4,m}(\rho; W) = (R_C/4)^{|m|/2}\rho^{1/2-|m|/2}\left(1 + O(\rho)\right),$$

$$C_{3,m}(\rho;W)=.\frac{\Gamma(\gamma_3)\Gamma(|m|)}{\Gamma(\alpha_1)\Gamma(\beta_1)}(R_C/4)^{|m|/2}\rho^{1/2-|m|/2}\left(1+O(\rho)\right),$$
$$\operatorname{Im}W>0 \text{ or } W=0.$$

3.3.3 Asymptotics, $\Delta = R_C - \rho \to 0$ ($\delta = 1 - x \to 0$)

We have

$$\delta=\frac{\Delta^2}{4R_C^2}(1+O(\Delta)),\ p_0(r)=\frac{4\Delta^2}{R_C^2}(1+O(\Delta)),$$
$$C_{3,m}(\rho;W)=2^{-3/2-2\nu}R_C^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),$$

$$C_{1,m}(\rho;W)=\frac{\Gamma(\gamma_1)\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}2^{-3/2+2\nu}R_C^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)),$$
$$\operatorname{Im}W>0 \text{ or } W=0.$$

3.3.4 Wronskian

$$\mathrm{Wr}(C_{1,m},C_{3,m})=-\frac{\Gamma(\gamma_1)\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)}=-\omega_m(W)=-|m|C_m(W).$$

We see that eq. (3.5) has no s.-integrable silutions for $\operatorname{Im}W>0$. This means that the deficiency indices of the symmetric operator (see below) are equal to $(0,0)$.

3.3.5 Symmetric operator $\hat{h}_m$

A symmetric operator $\hat{h}_m$ is defined in the Hilbert space $\mathfrak{h}_m = L^2\cdot(0,R_C)$ by equation

$$\hat{h}_m:\left\{\begin{array}{l} D_{h_m}=\mathcal{D}(0,R_C) \\ \hat{h}_m\psi_m=\check{h}_m\psi_m,\ \forall\psi_m\in D_{h_m}, \end{array}\right. , \tag{3.10}$$

the differential operation $\check{h}_m$ is given by eq. (3.4).

3.3.6 Adjoint operator $\hat{h}_m^+ = \hat{h}_m^*$

It is easy to prove by standard way that the adjoint operator $\hat{h}_m^+$ coincides with the operator $\hat{h}_m^*$,

$$\hat{h}_m^+:\left\{\begin{array}{l} D_{h_m^+}=D^*_{\check{h}_m}(0,R_C)=\{\psi_*,\psi_*'\text{ are a.c. in }(0,R_C), \\ \psi_*,\hat{h}_m^+\psi_*\in L^2(0,R_C)\} \\ \hat{h}_m^+\psi_*(\rho)=\check{h}_m\psi_*(\rho),\ \forall\psi_*\in D_{h_m^+} \end{array}\right. .$$

3.3.7 Asymptotics

Because $\check{h}_m\psi_* \in L^2(0,R_C)$, we have

$$\check{h}_m\psi_*(\rho) = \eta(\rho),\ \eta \in L^2(0,R_C),$$

and we can represent ψ_* in the form

$$\begin{aligned}
&\psi_*(\rho) = c_1 C_{1,m}(\rho;0) + c_2 C_{3,m}(\rho;0) + I(\rho), \\
&\psi_*'(\rho) = c_1 \partial_\rho C_{1,m}(\rho;0) + c_2 \partial_\rho C_{3,m}(\rho;0) + I'(\rho),
\end{aligned}$$

where

$$\begin{aligned}
I(\rho) &= \frac{C_{1,m}(\rho;0)}{\omega_m(0)} \int_\rho^{R_C} C_{3,m}(y;0)\eta(y)dy + \frac{C_{3,m}(\rho;0)}{\omega_m(0)} \int_0^\rho C_{1,m}(y;0)\eta(y)dy, \\
I'(\rho) &= \frac{\partial_\rho C_{1,m}(\rho;0)}{\omega_m(0)} \int_\rho^{R} C_{3,m}(y;0)\eta(y)dy + \frac{\partial_\rho C_{3,m}(\rho;0)}{\omega_m(0)} \int_0^\rho C_{1,m}(y;0)\eta(y)dy.
\end{aligned}$$

I) $\rho \to 0$

We obtain with the help of the Cauchy-Bunyakovskii inequality (CB-inequality):

$$I(\rho) = \left\{ \begin{array}{c} O(\rho^{3/2}),\ |m| \geq 3 \\ O(\rho^{3/2}\sqrt{\ln\rho}),\ |m| = 2 \end{array} \right.,\ I'(\rho) = \left\{ \begin{array}{c} O(\rho^{1/2}),\ |m| \geq 3 \\ O(\rho^{1/2}\sqrt{\ln\rho}),\ |m| = 2 \end{array} \right.,$$

such that we have

$$\begin{aligned}
&\psi_*(\rho) = c_2 c \rho^{1/2-|m|/2}\left(1 + O(\rho)\right) + \left\{ \begin{array}{c} O(\rho^{3/2}),\ |m| \geq 3 \\ O(\rho^{3/2}\sqrt{\ln\rho}),\ |m| = 2 \end{array} \right., \\
&c = (R_C/4)^{|m|/2} \frac{\Gamma(\gamma_3)\Gamma(2\mu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}.
\end{aligned}$$

The condition $\psi_* \in L^2(0,R_C)$ gives $c_2 = 0$, such that we find finally

$$\begin{aligned}
&\psi_*(\rho) = \left\{ \begin{array}{c} O(\rho^{3/2}),\ |m| \geq 3 \\ O(\rho^{3/2}\sqrt{\ln\rho}),\ |m| = 2 \end{array} \right.,\ \psi_*'(\rho) = \left\{ \begin{array}{c} O(\rho^{1/2}),\ |m| \geq 3 \\ O(\rho^{1/2}\sqrt{\ln\rho}),\ |m| = 2 \end{array} \right., \\
&[\psi_*, \chi_*]_0 = 0,\ \forall \psi_*, \chi_* \in D_{h_m^+}.
\end{aligned}$$

II) $\rho \to R_C$

In this case, we prove that $[\psi_*, \chi_*]^{R_C} = 0$, $\forall \psi_*, \chi_* \in D_{h_m^+}$. Indeed, consider the Hilbert space $\mathfrak{h}_{c,m} = L^2(c,R_C)$, c is an interior point of the interval $(0,R_C)$. and an symmetric operator $\hat{h}_{c,m}$, $D_{h_{c,m}} = \mathcal{D}(c,R_C)$, acting as $\check{h}_m$. We choose the functions $C_{1,m}(\rho;W)$ and $C_{3,m}(\rho;W)$ as the independent solutions of eq. (3.5) for $\operatorname{Im} W > 0$. The left end c of the interval $(0,R_C)$ is regular and both solutions $C_{1,m}$ and $C_{3,m}$ are s.-interable on the end c. The right end R_C is singular. On the right end R_C, the solution $C_{3,m}$ is s.-integrable, but $C_{1,m}$ is not. Thus, there is only one s. integrable solution of eq. (3.5) on the interval (c,R_C) for $\operatorname{Im} W > 0$ and the deficient indexes of the symmetric operator $\hat{h}_{c,m}$ are equal to $(1,1)$. In this case, according to [[7], Lemma on the page 213], we have $[\psi_*, \chi_*]^{R_C} = 0$, $\forall \psi_*, \chi_* \in D_{h_{c,m}^+}$. Because the restriction ψ_{c*} on the interval (c,R_C) of any function $\psi_* \in D_{h_m^+}$ belongs to $D_{h_{c,m}^+}$, $\psi_{c*} \in D_{h_{c,m}^+}$. $\forall \psi_* \in D_{h_m^+}$, we obtain that $[\psi_*, \chi_*]^{R_C} = 0$, $\forall \psi_*, \chi_* \in D_{h_m^+}$.

3.3.8 Self-adjoint hamiltonian $\hat{h}_{\mathfrak{e}m}$

Because $\omega_{h_m^+}(\chi_*, \psi_*) = \Delta_{h_m^+}(\psi_*) = 0$ (and also because $C_{1,m}(\rho; W)$ and $C_{3,r}(u; W)$ and any their linear combinations are not s.-integrable on the interval $(0, R_C)$ for $\operatorname{Im} W \neq 0$), the deficiency indices of initial symmetric operator $\hat{h}_m$ are zero, which means that $\hat{h}_{\mathfrak{e}m} = \hat{h}_m^+$ is a unique s.a. extension of the initial symmetric operator $\hat{h}_m$:

$$\hat{h}_{\mathfrak{e}m} : \begin{cases} D_{h_{\mathfrak{e}m}} = D^*_{\check{h}_m}(0, R_C) \\ \hat{h}_{\mathfrak{e}m}\psi_*(\rho) = \check{h}_m\psi_*(\rho), \ \forall\psi_* \in D_{h_{\mathfrak{e}m}} \end{cases} .$$

3.3.9 The guiding functional $\Phi(\xi; W)$

As a guiding functional $\Phi(\xi; W)$ we choose

$$\begin{aligned} &\Phi(\xi; W) = \int_0^{R_C} C_{1,m}(y; W)\xi(y)dy, \ \xi \in \mathbb{D} = D_r(0, R_C) \cap D_{h_{\mathfrak{e}m}}. \\ &D_r(0, R_C) = \{\xi(u) : \ \operatorname{supp}\xi \subseteq [0, \beta_\xi], \ \beta_\xi < R_C\}. \end{aligned}$$

The guiding functional $\Phi(\xi; W)$ is simple and the spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple.

3.3.10 Green function $G_m(\rho, y; W)$, spectral function $\sigma_m(E)$

We find the Green function $G_m(\rho, y; W)$ as the kernel of the integral representation

$$\psi(\rho) = \int_0^{R_C} G_m(\rho, y; W)\eta(y)dy, \ \eta \in L^2(0, R_C),$$

of unique solution of an equation

$$(\hat{h}_{\mathfrak{e}m} - W)\psi(\rho) = \eta(\rho), \ \operatorname{Im} W > 0, \tag{3.11}$$

for $\psi \in D_{h_{\mathfrak{e}m}}$. General solution of eq. (3.11) can be represented in the form

$$\begin{aligned} &\psi(\rho) = a_1 C_{1,m}(\rho; W) + a_3 C_{3,m}(\rho; W) + I(\rho), \\ &I(\rho) = \frac{C_{1,m}(\rho; W)}{\omega_m(W)} \int_\rho^{R_C} C_{3,m}(y; W)\eta(y)dy + \frac{C_{3,m}(\rho; W)}{\omega_m(W)} \int_0^\rho C_{1,m}(y; W)\eta(y)dy, \\ &I(\rho) = \begin{cases} O(\rho^{3/2}), \ |m| \geq 3 \\ O(\rho^{3/2}\sqrt{\ln\rho}), \ |m| = 2 \end{cases}, \ \rho \to 0, \ I(\rho) = O\left(\Delta^{-1/2}\right), \ \rho \to R_C. \end{aligned}$$

A condition $\psi \in L^2(0, \rho)$ gives $a_1 = a_3 = 0$, such that we find

$$\begin{aligned} &G_m(\rho, y; W) = \frac{1}{\omega_m} \begin{cases} C_{3,m}(\rho; W)C_{1,m}(y; W), \ \rho > y \\ C_{1,m}(\rho; W)C_{3,m}(y; W). \ \rho < y \end{cases} = \\ &= \pi\Omega_m(W)C_{1,m}(\rho; W)C_{1,m}(y; W) + \frac{1}{|m|} \begin{cases} C_{4,m}(\rho; W)C_{1,m}(y; W), \ \rho > y \\ C_{1,m}(\rho; W)C_{4,m}(y; W), \ \rho < y \end{cases}, \\ &\Omega_m(W) \equiv \frac{B_m(W)}{\pi\omega_m(W)} = \frac{(-1)^{|m|+1}[\psi(\alpha_1) + \psi(\alpha_2) + \psi(\beta_1) + \psi(\beta_2)]\mathcal{A}_m(W)}{2\pi\Gamma^2(\gamma_1)}, \\ &\mathcal{A}_m(W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)}. \end{aligned} \tag{3.12}$$

Note that the last term in the r.h.s. of eq. (3.12) is real for $W = E$. From the relation

$$[C_{1,m}(\rho_0; E)]^2 \sigma'_m(E) = \frac{1}{\pi} \operatorname{Im} G_m(\rho_0 - 0, \rho_0 + 0; E + i0),$$

where $f(E + i0) \equiv \lim_{\varepsilon \to +0} f(E + i\varepsilon)$, $\forall f(W)$, we find

$$\sigma'_m(E) = \operatorname{Im} \Omega_m(E + i0).$$

Consider $\Omega_m(W)$ in more details.

Using relations

$$\psi(\alpha_2) = \psi(\alpha_1) - T_{(\alpha)},\ T_{(\alpha)} = \sum_{k=1}^{|m|} \frac{1}{\alpha_1 - k},$$

$$\psi(\beta_2) = \psi(\beta_1) - T_{(\beta)},\ T_{(\beta)} = \sum_{k=1}^{|m|} \frac{1}{\beta_1 - k},$$

$$T_{(\alpha,\beta)}(W) = T_{(\alpha)} + T_{(\beta)} = \frac{2\nu \mathcal{B}_m(W)}{\mathcal{A}_m(W)},$$

where $\mathcal{B}_m(W)$ is an polinomial in ν and σ, even in both ν and σ, and therefore, is a real-entire polynomial in W, we can represent $\Omega_m(W)$ in the form

$$\Omega_m(W) = \Omega_{1m}(W) + \Omega_{2m}(W),$$

$$\Omega_{1m}(W) = \frac{(-1)^{|m|+1} \mathcal{A}_m(W)}{\pi \Gamma^2(\gamma_1)} \left[\psi(\alpha_1) + \psi(\beta_1)\right],$$

$$\Omega_{2m}(W) = -\nu \tilde{\Omega}_{2m}(W),\ \tilde{\Omega}_{2m}(W) = \frac{(-1)^{|m|+1} \mathcal{B}_m(W)}{\pi \Gamma^2(\gamma_1)}.$$

3.3.11 Spectrum

3.3.12 $W = (1 + 4R_C g)/R_C^2 + \tilde{\Delta}$, $\tilde{\Delta} \sim 0$

A direct estimation gives

$$\Omega_m(W) = \begin{cases} \Omega_m(W_0) + O(\sqrt{\tilde{\Delta}}),\ g \neq g_{m,k} \\ O(1/\sqrt{\tilde{\Delta}}),\ g = g_{m,k}, \end{cases},$$

$R_C g_{m,k} = -N_{m,k}^2$, $N_{m,k} = 1 + |m| + 2k, W_0 = (1 + 4R_C g)/R_C^2, \operatorname{Im}[\Omega_m(W_0)] = 0$. This result means that the levels with $E = E_0$ are absent.

3.3.13 $w = R_C^2 E > 1 + 4R_C g$

In this case, we have $\alpha_1 = 1/2 + |m|/2 + \sigma - i\sqrt{w - 1 - 4R_C g}/4$, $\beta_1 = 1/2 + |m|/2 - \sigma - i\sqrt{w - 1 - 4R_C g}/4$ and it is easy to prove that $\alpha_1, \beta_1 \notin \mathbb{Z}_-$, such that $\Omega_m(E)$ is finite complex function of E and we have

$$\sigma'_m(E) = \operatorname{Im} \Omega_m(E) \equiv \varrho_m^2(E) > 0.$$

The spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple and continuous, $\operatorname{spec}\hat{h}_{\mathfrak{e}m} = [(1 + 4R_C g)/R_C^2, \infty)$. Note that

$$\sigma'_m(E)|_{\Delta \to +0} = \begin{cases} O(\sqrt{\tilde{\Delta}}),\ g \neq g_{m,k} \\ O(1/\sqrt{\tilde{\Delta}}),\ g = g_{m,k} \end{cases}, \qquad (3.13)$$

$\Delta = E - (1 + 4R_C g)/R_C^2 \to +0$.

3.3.14 $w = R_C^2 E \le 1 + 4R_C g$

$E < 1/R_C^2$ In this case, we have $\operatorname{Im}\nu|_{W=E} = \operatorname{Im}\sigma|_{W=E} = \operatorname{Im}\alpha_1|_{W=E} = \operatorname{Im}\beta_1|_{W=E} = 0$, $\operatorname{Im}\psi(\alpha_1)|_{W=E} = 0$, and

$$\sigma'_m(E) = \frac{(-1)^{|m|+1}\mathcal{A}_m(E)}{\pi\Gamma^2(\gamma_1)} \operatorname{Im}\psi(\beta_1)|_{W=E+i0}\,.$$

$\sigma'_m(E)$ can be different from zero in the points $E_{m,n}$ when $\beta_1 = -n$, $n = 0, 1, 2, ...$, i.e., $E_{m,n}$ satisfy the equations

$$\sqrt{1 - w_{m,n}} = \sqrt{1 - w_{m,n} + 4R_C g} + 2N_{m,n},\ N_{m,n} = 1 + |m| + 2n. \tag{3.14}$$

Eq. (3.14) has solutions only if $g < 0$ and only one inequality

$$w_{m,n} \le 1 - 4R_C|g| \tag{3.15}$$

must be satisfied It follows from eq.(3.14) that

$$N_{m,n}\sqrt{1 - w_{m,n} - 4R_C|g|} = R_C|g| - N_{m,n}^2, \tag{3.16}$$

and we obtain one more inequality

$$R_C|g| \ge N_{m,n}^2. \tag{3.17}$$

It follows from eq.(3.16) that

$$\begin{aligned}
&w_{m,n} = 1 - (R_C|g|/N_{m,n} + N_{m,n})^2 = \\
&\ = 1 - 4R_C|g| - R_C|g|\left[(a + a^{-1})^2 - 4\right], \\
&E_{m,n} = w_{m,n}/R_C^2,\ a = N_{m,n}/\sqrt{R_C|g|},
\end{aligned}$$

so that inequality (3.15) is satisfied. It follows from inequality (3.17) and eq. (3.13) that there are no levels for $g \ge g_{m,0} = -N_{m,0}^2/R_C = -(1+|m|)^2/R_C$ and there are $n_{\max}+1$ levels $n = 0, 1, ..., n_{\max} = k$ for $\sqrt{R_C|g|} = 1 + |m| + 2(k + \delta)$, $k = 0, 1, ...,$ $0 < \delta \le 1$. Note that $1 - 4R_C|g| > E_{m,n} > E_{m,n-1}$, $n = 1, ..., n_{\max}$.

We have

$$\sigma'_m(E) = \sum_{n=0}^{n_{\max}} Q_{m,n}^2 \delta(E - E_{m,n}),\ Q_{m,n} = \sqrt{\frac{(-1)^{|m|}4\mathcal{A}_m(E_{m,n})(R_C^2 g^2 - N_{m,n}^4)}{\Gamma^2(\gamma_1)N_{m,n}^3 R_C^2}}..$$

The discrete part of the spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple and has the form

$$\operatorname{spec}\hat{h}_{\mathfrak{e}m} = \{E_{m,n},\ E_{m,n} < 1 - 4R_C|g|,\ n = 0, 1, ..., n_{\max}\}.$$

The discrete part of the spectrum is absent for $g \ge g_{m,0}$.

$w = 1$, $\sigma = 0$ We have in this case for $W = E = R_C^{-2}$: $g \ge 0$, $\alpha_1 = \beta_1 = 1/2 + |m|/2 + \sqrt{R_C g}/2$, $\operatorname{Im}\nu = 0$, $\operatorname{Im}\alpha_1 = 0$, $\alpha_1 > 0$, , and $\sigma'_m(R_C^{-2}) = 0$.

$w > 1$, $\sigma = i\varkappa/4$, $\varkappa = \sqrt{w-1}$ In this case, we have for $W = E$:$\mathrm{Im}\,\nu = 0$, $\alpha_{1,2} = 1/2 \pm |m|/2 + \nu + i\varkappa/4$, $\beta_{1,2} = 1/2 \pm |m|/2 + \nu - i\varkappa/4 = \overline{\alpha_1}$, such that $[\mathrm{Im}\,\psi(\alpha_{1,2}) + \mathrm{Im}\,\psi(\beta_{1,2})]_{W=E} = 0$, and

$$\sigma_m'(E) = 0.$$

Finally, we find for fixed m, $|m| \geq 2$:

The spectrum of $\hat{h}_{\mathfrak{e}m}$ is simple, $\mathrm{spec}\hat{h}_{\mathfrak{e}m} = [1 + 4R_C g)/R_C^2, \infty) \cup \{E_{m,n},\ n = 0, 1, ...n_{\max}\}$, the discrete part of spectrum is present for $g < g_{m,0}$. The set of functions

$$\left\{U_m(\rho; E) = \varrho_m(E) C_{1m}(\rho; E),\ E \geq 1 + 4R_C g)/R_C^2;\ U_{m,n}(\rho) = Q_{m,n} C_{1m}(\rho; E_{m,n}),\ n = 0, 1, ...n_{\max}\right\}$$

forms a complete orthogonalized system in $L^2(0, R_C)$.

3.4 $m = 1$

3.4.1 Useful solutions, $\rho < R_C$

We construct again the solutions of eq. (3.5) with the help of correspondence formulas (3.7), (3.8), and (3.9). We have

$$\begin{aligned}
&C_{1,1}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{1/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; 2; x) = \\
&(\text{for}\quad \mathrm{Im}\,W > 0) = Q_1(W) C_{3,1}(\rho; W) + Q_2(W) v_1(\rho; W), \\
&Q_1(W) = \frac{\Gamma(-2\nu)}{\Gamma(\alpha_4)\Gamma(\beta_4)},\ Q_2(W) = \frac{\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}, \\
&v_1(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{1/2}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_4, \beta_4; \gamma_4; 1-x),
\end{aligned}$$

$$\begin{aligned}
&C_{4,1}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}\frac{R^2 - r^2}{R^2 r^{1/2}} O_{4,1}^{<}(r; W) = \\
&= \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}(1-x)^{1/4+\nu}\lim_{\delta\to 0}[x^{-\mu_\delta}\mathcal{F}(\alpha_{2\delta}, \beta_{2\delta}; \gamma_{2\delta}; x) - \\
&- A_{1,\delta}(W)\Gamma(\gamma_{2\delta}) x^{\mu_\delta}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_{1\delta}; x)], \\
&A_{1,\delta}(W) = \frac{\Gamma(\alpha_1)\Gamma(\beta_1)}{\Gamma(\alpha_2)\Gamma(\beta_2)\Gamma(\gamma_{1\delta})},
\end{aligned}$$

$$\begin{aligned}
&C_{3,1}(r; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2} x^{1/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; \gamma_3; 1-x) = \\
&= B_1(W) C_{1,1}(\rho; W) + C_1(W) C_{4,1}(\rho; W),\ C_1(W) = \frac{\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)}, \\
&B_1(W) = C_1(W) f_1(W),\ f_1(W) = \frac{R_C g}{4}\left[\psi(\alpha_1) + \psi(\beta_1)\right] - \nu,
\end{aligned}$$

$$\begin{aligned}
&\alpha_{1,2\delta} = 1/2 \pm (1+\delta)/2 + \nu + \sigma,\ \beta_{1,2\delta} = 1/2 \pm (1+\delta)/2 + \nu - \sigma, \\
&\gamma_{1,2\delta} = 1 \pm (1+\delta) \\
&\alpha_{1,2} = 1/2 \pm 1/2 + \nu + \sigma,\ \beta_{1,2} = 1/2 \pm 1/2 + \nu - \sigma, \\
&\alpha_4 = 1 - \nu + \sigma,\ \beta_4 = 1 - \nu - \sigma,\ \gamma_{3,4} = 1 \pm 2\nu.
\end{aligned}$$

$C_{1,1}(\rho; W)$ and $C_{4,1}(\rho; W)$ are real-entire in W.

3.4.2 Asymptotics, $\rho \to 0$ $(x \to 0)$

We have

$$
\begin{aligned}
C_{1,1}(\rho;W) &= C_{1,1\text{as}}(\rho)(1+O(\rho)), \\
C_{4,1}(\rho;W) &= C_{4,1\text{as}}(\rho)\left(1+O(\rho\ln\rho)\right),
\end{aligned}
$$

$$
\begin{aligned}
& C_{3,1}(\rho;W) = C_1(W)[f_1(W)C_{1,1\text{as}}(\rho) + C_{4,1\text{as}}(\rho)](1+O(\rho\ln\rho), \\
& \operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

$$
\begin{aligned}
C_{1,1\text{as}}(\rho) &= (4/R_C)^{1/2}\rho, \\
C_{4,1\text{as}}(\rho) &= (R_C/4)^{1/2}\left[1 + g\rho\left(\ln(4\rho/R_C) + 2\mathbf{C}\right)\right]
\end{aligned}
$$

3.4.3 Asymptotics, $\Delta = R_C - \rho \to 0$ $(\delta = 1 - x \to 0)$

We have

$$
C_{3,1}(\rho;W) = 2^{-3/2-2\nu}R_C^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),
$$

$$
\begin{aligned}
& C_{1,1}(\rho;W) = \frac{\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}2^{-3/2+2\nu}R_C^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)), \\
& \operatorname{Im} W > 0 \text{ or } W = 0.
\end{aligned}
$$

3.4.4 Wronskians

$$
\begin{aligned}
& \operatorname{Wr}(C_{1,1}, C_{4,1}) = -1, \\
. & \operatorname{Wr}(C_{1,1}, C_{3,1}) = -\frac{\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)} = -\omega_1(W) = -C_1(W)
\end{aligned}
$$

We see that any solutions of eq. (3.5) are s.-integrable at $\rho = 0$ and there is one (and only one) silution, $C_{3,1}$, which is s.-integrable on the interval $(0, R_C)$ for $\operatorname{Im} W > 0$. This means that the deficiency indices of the symmetric operator (see below) are equal to $(1,1)$.

3.4.5 Symmetric operator $\hat{h}_1$

A symmetric operator $\hat{h}_1$ is defined by eq. (3.10) with $m = 1$.

3.4.6 Adjoint operator $\hat{h}_1^+ = \hat{h}_1^*$

It is easy to prove by standard way that the adjoint operator $\hat{h}_1^+$ coincides with the operator $\hat{h}_1^*$,

$$
\hat{h}_1^+ : \left\{
\begin{array}{l}
D_{h_1^+} = D^*_{\check{h}_1}(0, R_C) = \{\psi_*, \psi'_* \text{ are a.c. in } (0, R_C), \\
\psi_*, \hat{h}_1^+\psi_* \in L^2(0, R_C)\} \\
\hat{h}_1^+\psi_*(\rho) = \check{h}_1\psi_*(\rho), \ \forall \psi_* \in D_{h_1^+}
\end{array}
\right. .
$$

3.4.7 Asymptotics

Because $\check{h}_1\psi_* \in L^2(0, R_C)$, we have

$$\check{h}_1\psi_*(\rho) = \eta(\rho),\ \eta \in L^2(0, R_C),$$

and we can represent ψ_* in the form

$$\begin{aligned}
\psi_*(\rho) &= c_1 C_{1,1}(\rho; 0) + c_2 C_{4,1}(\rho; 0) + I(\rho), \\
\psi'_*(\rho) &= c_1 \partial_\rho C_{1,1}(\rho; 0) + c_2 \partial_\rho C_{4,1}(\rho; 0) + I'(\rho),
\end{aligned}$$

where

$$\begin{aligned}
I(\rho) &= C_{4,1}(\rho; 0) \int_0^\rho C_{1,1}(y; 0)\eta(y)dy - C_{1,1}(\rho; 0) \int_0^\rho C_{4,1}(y; 0)\eta(y)dy, \\
I'(\rho) &= \partial_\rho C_{4,1}(\rho; 0) \int_0^\rho C_{1,1}(y; 0)\eta(y)dy - \partial_\rho C_{1,1}(\rho; 0) \int_0^\rho C_{4,1}(y; 0)\eta(y)dy.
\end{aligned}$$

I) $\rho \to 0$

We obtain with the help of the Cauchy-Bunyakovskii inequality (CB-inequality):

$$I(\rho) = O(\rho^{3/2}),\ I'(\rho) = O(\rho^{1/2}),$$

such that we have

$$\begin{aligned}
\psi_*(\rho) &= c_1 C_{1,1\mathrm{as}}(\rho) + c_2 C_{4,1\mathrm{as}}(\rho) + O(\rho^{3/2}), \\
\psi'_*(\rho) &= c_1 C'_{1,1\mathrm{as}}(\rho) + c_2 C'_{4,1\mathrm{as}}(\rho) + O(\rho^{1/2}).
\end{aligned}$$

II) $\rho \to R_C$

In this case, we prove that $[\psi_*, \chi_*]^{R_C} = 0$, $\forall \psi_*, \chi_* \in D_{h_m^+}$, such that we obtain

$$\Delta_{h_1^+} = \overline{c_2}c_1 - \overline{c_1}c_2) = \frac{i}{2}(\overline{c_+}c_+ - \overline{c_-}c_-),\ c_\pm = c_1 \pm ic_2.$$

3.4.8 Self-adjoint hamiltonians $\hat{h}_{1\zeta}$

The condition $\Delta_{h_1^+}(\psi) = 0$ gives

$$\begin{aligned}
&c_- = -e^{2i\zeta}c_+,\ |\zeta| \leq \pi/2,\ \zeta = -\pi/2 \sim \zeta = \pi/2,\ \Longrightarrow \\
&\Longrightarrow\ c_1 \cos\zeta = c_2 \sin\zeta,
\end{aligned}$$

or

$$\begin{aligned}
&\psi(\rho) = C\psi_{\zeta\mathrm{as}}(\rho) + O(\rho^{3/2}),\ \psi'(\rho) = C\psi'_{\zeta\mathrm{as}}(\rho) + O(\rho^{1/2}), \\
&\psi_{\zeta\mathrm{as}}(\rho) = C_{1,1\mathrm{as}}(\rho)\sin\zeta + C_{4,1\mathrm{as}}(\rho)\cos\zeta.
\end{aligned} \tag{3.18}$$

Thus we have a family of s.a. hamiltohians $\hat{h}_{1\zeta}$,

$$\hat{h}_{1\zeta} : \begin{cases} D_{h_{1\zeta}} = \{\psi \in D_{h_1^+},\ \psi \text{ satisfy the boundary condition (3.18)} \\ \hat{h}_{1\zeta}\psi = \check{h}_1\psi,\ \forall\psi \in D_{h_{1\zeta}} \end{cases}. \tag{3.19}$$

3.4.9 The guiding functional

As a guiding functional $\Phi_{1\zeta}(\xi;W)$ we choose

$$\Phi_{1\zeta}(\xi;W)=\int_0^{R_C}U_{1\zeta}(\rho;W)\xi(\rho)d\rho,\ \xi\in\mathbb{D}_{1\zeta}=D_r(0,R_C)\cap D_{h_{1\zeta}}.$$
$$.U_{1\zeta}(\rho;W)=C_{1,1}(\rho;W)\sin\zeta+C_{4,1}(\rho;W)\cos\zeta,$$

$U_{1\zeta}(\rho;W)$ is real-entire solution of eq. (3.5) satisfying the boundary condition (3.18).

The guiding functional $\Phi_{1\zeta}(\xi;W)$ is simple and the spectrum of $\hat{h}_{1\zeta}$ is simple.

3.4.10 Green function $G_{1\zeta}(\rho,y;W)$, spectral function $\sigma_{1\zeta}(E)$

We find the Green function $G_{1\zeta}(\rho,y;W)$ as the kernel of the integral representation

$$\psi(\rho)=\int_0^{\infty}G_{1\zeta}(\rho,y;W)\eta(y)dy,\ \eta\in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{1\zeta}-W)\psi(\rho)=\eta(\rho),\ \operatorname{Im}W>0, \tag{3.20}$$

for $\psi\in D_{h_{1\zeta}}$. General solution of eq. (3.20) (under condition $\psi\in L^2(0,R)$) can be represented in the form

$$\psi(\rho)=aC_{3,1}(\rho;W)+\frac{C_{1,1}(\rho;W)}{C_1(W)}\eta_3(W)+I(\rho),\ \eta_3(W)=\int_0^R C_{3,1}(y;W)\eta(y)dy,$$
$$I(\rho)=\frac{C_{3,1}(\rho;W)}{C_1(W)}\int_0^\rho C_{1,1}(y;W)\eta(y)dy-\frac{C_{1,1}(\rho;W)}{C_1(W)}\int_0^\rho C_{3,1}(y;W)\eta(y)dy,$$
$$I(\rho)=O\left(\rho^{3/2}\right),\ \rho\to 0.$$

A condition $\psi\in D_{h_{1\zeta}}$, (i.e.ψ satisfies the boundary condition (3.18)) gives

$$a=-\frac{\cos\zeta}{C_1^2(W)\omega_\zeta(W)}\eta_3(W),\ \omega_\zeta(W)=f_1(W)\cos\zeta-\sin\zeta,$$

$$\begin{gathered}G_{1\zeta}(\rho,y;W)=\pi\Omega_{1\zeta}(W)U_{1\zeta}(\rho;W)U_{1\zeta}(y;W)-\\ -\left\{\begin{array}{l}\tilde{U}_{1\zeta}(\rho;W)U_{1\zeta}(y;W),\ \rho>y\\ U_{1\zeta}(\rho;W)\tilde{U}_{1\zeta}(y;W),\ \rho<y\end{array}\right.,\\ \Omega_{1\zeta}(W)=-\frac{\tilde{\omega}_\zeta(W)}{\pi\omega_\zeta(W)},\ \tilde{\omega}_\zeta(W)=f_1(W)\sin\zeta+\cos\zeta,\\ U_{1\zeta}(y;W)=C_{1,1}(\rho;W)\sin\zeta+C_{4,1}(\rho;W)\cos\zeta,\\ \tilde{U}_{1\zeta}(\rho;W)=C_{1,1}(\rho;W)\cos\zeta-C_{4,1}(\rho;W)\sin\zeta,\end{gathered} \tag{3.21}$$

where we used an equality

$$C_{3,1}(\rho;W)=\tilde{\omega}_\zeta(W)U_\zeta(\rho;W)+\omega_\zeta(W)\tilde{U}_{1\zeta}(\rho;W).$$

Note that the functions $U_{1\zeta}(\rho;W)$ and $\tilde{U}_{1\zeta}(\rho;W)$ are real-entire in W and the last term in the r.h.s. of eq. (3.21) is real for $W=E$. For $\sigma'_{1\zeta}(E)$, we find

$$\sigma'_{1\zeta}(E)=\operatorname{Im}\Omega_{1\zeta}(E+i0).$$

3.4.11 Spectrum

3.4.12 $W=(1+4R_Cg)/R_C^2+\tilde{\Delta}$, $\tilde{\Delta}\sim 0$

A direct estimation gives

$$\Omega_{1\zeta}(W)=\left\{\begin{array}{l}\Omega_{1\zeta}(W_0)+O(\sqrt{\tilde{\Delta}}),\ g=g_{1k}\text{ or }g\neq g_{1k},\ \zeta\neq\zeta_1\\ O(1/\sqrt{\tilde{\Delta}}),\ g\neq g_{1k},\ \zeta=\zeta_1\end{array}\right.,$$

$R_Cg_{1,k}=-N_{1,k}^2$, $N_{1,k}=2(1+k)$, $\mathrm{Im}[\Omega_{1\zeta}(W_0)]=0, \tan\zeta_1=f_{1|0}=f_1(W)|_{\tilde{\Delta}=0}$. This result means that the levels with $E=E_0$ are absent.

3.4.13 $\zeta=\pi/2$

First we consider the case $\zeta=\pi/2$.

In this case, we have $U_{1\pi/2}(\rho;W)=C_{1,1}(\rho;W)$ and

$$\sigma'_{1\pi/2}(E)=\frac{1}{\pi}\,\mathrm{Im}\ f_1(W)|_{W=E+i0}\,.$$

All results for spectrum ang spectral function can be obtained from the corresponding results of previous section by setting there $m=1$.

$w=R_C^2E>1+4R_Cg$ In this case, $f_1(E)$ is a finite complex function and we find

$$\sigma'_{1\pi/2}(E)=\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{V}_1(E)\equiv\varrho^2_{1\pi/2}(E)>0.$$

where $f_1(E)=\mathcal{U}_1(E)+i\mathcal{V}_1(E), \mathcal{U}_1(E)=\mathrm{Re}\,f_1(E), \mathcal{V}_1(E)=\mathrm{Im}\,f_1(E)>0$. The spectrum of $\hat{h}_{1\pi/2}$ is simple and continuous, $\mathrm{spec}\hat{h}_{1\pi/2}=[(1+4R_Cg)/R_C^2,\infty)$, and

$$\sigma'_{1\pi/2}(E)=\left\{\begin{array}{c}O(\sqrt{\Delta}),\ g\neq g_{1,k}\\ O(1/\sqrt{\Delta}),\ g=g_{1,k}\end{array}\right.,\ \Delta\to+0,$$

$\Delta=E-(1+4R_Cg)/R_C^2$.

$w=R_C^2E\leq 1+4R_Cg$, $w<1$ In this case, we have

$$\sigma'_{1\pi/2}(E)=\sum_{n=0}^{n_{\max}}Q^2_{1\pi/2,n}\delta(E-\mathcal{E}_{1,n}),\ Q_{1\pi/2,n}=\sqrt{\frac{|g|(R_C^2g^2-N_{1,n}^4)}{N_{1,n}^3R_C}},$$
$$\mathcal{E}_{1,n}=.\frac{1-(R_C|g|/N_{1,n}+N_{1,n})^2}{R_C^2},\ R_Cg<R_Cg_{1,0}=-N_{1,0}^2=-4.$$

The spectrum of $\hat{h}_{1\pi/2}$ is discrete and simple and has the form

$$\mathrm{spec}\hat{h}_{1\pi/2}=\{\mathcal{E}_{1,n},\ \mathcal{E}_{1,n}<1-4R_C|g|,\ n=0,1,...,n_{\max}\},$$

$n_{\max}=k$ for $\sqrt{R_C|g|}=2(1+k+\delta)$, $0<\delta\leq 1$. The discrete part of the spectrum is absent for $g\geq g_{1,0}=-N_{1,0}^2/R_C=-4/R_C$.

$w = 1$, $\sigma = 0$ We have in this case for $W = E = R_C^{-2}$: $g \geq 0$, $\alpha_1 = \beta_1 = 1 + \sqrt{R_C g}/2$, $\operatorname{Im}\nu = 0$, $\operatorname{Im}\alpha_1 = 0$, $\alpha_1 > 0$, , and $\sigma'_m(R_C^{-2}) = 0$.

$w > 1$, $\sigma = i\varkappa/4$, $\varkappa = \sqrt{w-1}$ In this case, we have for $W = E$:$\operatorname{Im}\nu = 0$, $\alpha_1 = 1+\nu+i\varkappa/4$, $\beta_1 = 1+\nu-i\varkappa/4 = \overline{\alpha_1}$, such that $[\operatorname{Im}\psi(\alpha_1) + \operatorname{Im}\psi(\beta_1)]_{W=E} = 0$, and

$$\sigma'_m(E) = 0.$$

Finally, we find.

The spectrum of $\hat{h}_{1\pi/2}$ is simple, $\operatorname{spec}\hat{h}_{1\pi/2} = [1+4R_C g)/R_C^2, \infty) \cup \{\mathcal{E}_{1,n},\ n = 0, 1, ...n_{\max}\}$, the discrete part of spectrum is present for $g < g_{1,0}$. The set of functions

$$U_{1\pi/2}(\rho; E) = \varrho_{1,\pi/2}(E) C_{1,1}(\rho; E),\ E \geq 1 + 4R_C g)/R_C^2;$$
$$U_{1\pi/2,n}(\rho) = Q_{1\pi/2,n} C_{1,1}(\rho; \mathcal{E}_{1,n}),\ n = 0, 1, ...n_{\max}$$

forms a complete orthogonalized system in $L^2(0, R_C)$.

The same results for spectrum and eigenfunctions are obtained for the case $\zeta = -\pi/2$.

3.4.14 $|\zeta| < \pi/2$

Now we consider the case $|\zeta| < \pi/2$.

In this case, we can represent $\sigma'_{1\zeta}(E)$ in the form

$$\sigma'_{1\zeta}(E) = -\frac{1}{\pi\cos^2\zeta}\operatorname{Im}\frac{1}{f_{1\zeta}(E+i0)},\ f_{1\zeta}(W) = f_1(W) - \tan\zeta.$$

$E > (1 + 4R_C g)/R_C^2$ In this case, we have

$$\sigma'_{1\zeta}(E) = \frac{1}{\pi}\frac{\mathcal{V}_1(E)}{[\mathcal{U}_1(E)\cos\zeta - \sin\zeta]^2 + \mathcal{V}_1^2(E)\cos^2\zeta} \equiv \rho_{0\zeta}^2(E).$$

The spectrum of $\hat{h}_{1\zeta}$ is simple and continuous, $\operatorname{spec}\hat{h}_{1\zeta} = [(1 + 4R_C g)/R_C^2, \infty)$.

$w = 1 + 4R_C g + \Delta$, $\Delta \sim 0$ In this case, we have

$$4\nu = \sqrt{-\Delta} = \left\{\begin{array}{l} -i\sqrt{\Delta},\ \Delta \geq 0 \\ \sqrt{|\Delta|},\ \Delta \leq 0 \end{array}\right.,\ \sigma = \left\{\begin{array}{l} -i\sqrt{R_C g}/2 + O(\Delta),\ g > 0 \\ \nu,\ g = 0 \\ \sqrt{R_C|g|}/2 + O(\Delta),,\ g < 0 \end{array}\right.,$$

$$\alpha_1 = 1 + \left\{\begin{array}{l} -i\sqrt{R_C g}/2 + \nu + O(\Delta),\ g > 0 \\ 2\nu,\ g = 0 \\ \sqrt{R_C|g|}/2 + \nu + O(\Delta),,\ g < 0 \end{array}\right.,$$

$$\beta_1 = 1 + \left\{\begin{array}{l} i\sqrt{R_C g}/2 + \nu + O(\Delta),\ g > 0 \\ 0,\ g = 0 \\ -\sqrt{R_C|g|}/2 + \nu + O(\Delta),,\ g < 0 \end{array}\right..$$

A direct estimation gives

$$\sigma'_{1\zeta}(E) = \left\{\begin{array}{l} \left\{\begin{array}{l} O(\sqrt{\Delta}),\ g = g_{1k} \text{ or } g \neq g_{1k},\ \zeta \neq \zeta_1 \\ O(1/\sqrt{\Delta}),\ g \neq g_{1k},\ \zeta = \zeta_1 \end{array}\right.,\ \Delta > 0 \\ 0,\ \Delta < 0 \end{array}\right.,$$

where $\tan\zeta_1 = f_{1|0} = f_1(E)|_{\Delta=0}$. Note that the discrete level with $E = (1 + 4R_C g)R_C^2$ is absent.

$E < (1+4R_C g)R_C^2$ In this case, we have $\operatorname{Im}\nu|_{W=E} = 0,\ \nu|_{W=E} > 0$,

$$\alpha_1 = 1+\nu - i\sqrt{w-1}/4,\ \beta_1 = \overline{\alpha_1},\ w \geq 1,$$
$$\alpha_1 = 1+\nu + \sqrt{1-w}/4,\ \beta_1 = 1+\nu - \sqrt{1-w}/4,\ w < 1.$$

Thus, we have: the function $[f_{1\zeta}(E)]^{-1}$ is real except the points $E_{1n}(\zeta)$,

$$f_{1\zeta}(E_{1n}(\zeta)) = 0, \tag{3.22}$$

such that we obtain

$$\sigma'_{1\zeta}(E) = \sum_{n\in\mathcal{N}_1} Q^2_{1\zeta,n}\delta(E - E_{1n}(\zeta)),\ Q_{1\zeta,n} = \frac{1}{\sqrt{\partial_E f_{1\zeta}(E_{1n}(\zeta))}\cos\zeta},$$
$$\partial_E f_1(E) > 0,$$

where $\mathcal{N}_1$ is a subset of $\mathbb{Z}$ to be described below. Furthermore, we find

$$\partial_\zeta E_{1n}(\zeta) = [\partial_E f_1(E_n(\zeta))\cos^2\zeta]^{-1} > 0.$$

$g \geq g_{1,0} = -4/R_C$ In this case, the function $f_1(E)$ has the properties: $f_1(E) \to -\infty$ as $E \to -\infty$; $f_1(E) \to \tan\zeta_1 - 0 = f_{1|0} - 0$ as $E \to (1+4R_C g)R_C^2 - 0$;.$f_1(E)$ increases monotonocally on the interval $(-\infty, (1+4R_C g)R_C^2)$. Then we find: in the interval $(-\infty, (1+4R_C g)R_C^2)$, for any fixed $\zeta \in (-\pi/2, \zeta_1)$, there is one level $E_{1,-1}(\zeta)$ which run monotonocally from $-\infty$ to $(1+4R_C g)R_C^2 - 0$ as ζ run from $-\pi/2 - 0$ to $\zeta_1 - 0$; there are no discrete levels on the interval $(-\infty, (1+4R_C g)R_C^2)$ for $\zeta \in (\zeta_1, \pi/2)$. Formally, eq. (3.22) has solution $E_{1,-1}(\zeta_1) = (1+4R_C g)R_C^2$ for $\zeta = \zeta_1$. However, as was noted above, such levels are absent We find also

$$\mathcal{N}_1 = \mathcal{N}_{1,-1}(\zeta) = \left\{ \begin{array}{l} \varnothing,\ \zeta \in [\zeta_1, \pi/2) \\ \{-1\},\ \zeta \in (-\pi/2, \zeta_1) \end{array} \right. .$$

$g_{1,k+1} \leq g < g_{1,k}$, $\sqrt{R_C|g|} = 2(1+k+\delta)$, $0 < \delta \leq 1$, $k \in \mathbb{Z}_+$ In this case, the function $f_1(E)$ has the properties: $f_1(E) \to -\infty$ as $E \to -\infty$; $f(\mathcal{E}_{1n} \pm 0) = \mp\infty$ $n = 0, 1, ..., n_{\max} = k$; $f_1(E) \to \tan\zeta_1 - 0 = f_{1|0}(E) - 0$ as $E \to (1+4R_C g)/R_C^2 - 0$. We find: in each interval $(\mathcal{E}_{1n-1}, \mathcal{E}_{1n})$, $n = 0, ..., k$ (we set $\mathcal{E}_{1,-1} = -\infty$), for any fixed $\zeta \in (-\pi/2, \pi/2)$, there is one level $E_{1n}(\zeta)$ which run monotonocally from $\mathcal{E}_{n-1} + 0$ to $\mathcal{E}_n - 0$ as ζ run from $-\pi/2 + 0$ to $\pi/2 - 0$; in the interval $(\mathcal{E}_k, (1+4R_C g)R_C^2)$; for any fixed $\zeta \in (-\pi/2, \zeta_1)$, there is one level $E_{k+1}(\zeta)$ which run monotonocally from $\mathcal{E}_k + 0$ to $(1+4R_C g)/R_C^2 - 0$ as ζ run from $-\pi/2 + 0$ to $\zeta_1 - 0$; thare are no levels in the interval $(\mathcal{E}_k, (1+4R_C g)/R_C^2)$ for $\zeta \in (\zeta_1, \pi/2)$. Formally, eq. (3.22) has solution $E_{1,k+1}(\zeta_1) = (1+4R_C g)/R_C^2$ for $\zeta = \zeta_1$. However, as was noted above, such levels are absent.. We find also

$$\mathcal{N}_1 = \mathcal{N}_{1,k}(\zeta) = \left\{ \begin{array}{l} \{0, 1, ..., k\},\ \zeta \in [\zeta_1, \pi/2) \\ \{0, 1, ..., k+1\},\ \zeta \in (-\pi/2, \zeta_1) \end{array} \right. .$$

Finally, we obtain. The spectrum of $\hat{h}_{1\zeta}$ is simple and $\operatorname{spec}\hat{h}_{1\zeta} = [(1+4R_C g)/R_C^2, \infty) \cup \{E_{1n}(\zeta),\ n \in \mathcal{N}_1\}$. The set of functions

$$\left\{U_{1\zeta,E}(\rho) = \varrho_{1\zeta}(E)U_{1\zeta}(\rho; E),\ E \geq (1+4R_C g)/R_C^2;\ U_{1\zeta,n}(\rho) = Q_{1\zeta,n}U_{1\zeta}(\rho; E_{1n}(\zeta),),\ n \in \mathcal{N}\right\}$$

forms a complete orthogonalized system in $L^2(0, R_C)$.

3.5 $m = -1$

Only modification which we must do is the following: the extension parameter for the case $m = -1$ should be considered as indendent of the extension parameter for the case $m = 1$. It is convenient to denote the extension parameter for the case $m = 1$ as $\zeta_{(1)}$ and for the case $m = -1$ as $\zeta_{(-1)}$.

3.6 $m = 0$

In this case, we have $\mu = 0$; $\alpha_1 = 1/2 + \nu + \sigma$, $\beta_1 = 1/2 + \nu - \sigma$.

3.6.1 Useful solutions

We need solutions of an equation

$$(\check{h}_0 - W)\psi_0^<(\rho) = 0, \tag{3.23}$$

where $\check{h}_0$ is given by eq. (3.2) with $m = 0$.

We use the following solution of eq.(3.23)

$$\begin{aligned}
&C_{1,0}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; 1; x) = \\
&(\text{for } \operatorname{Im} W > 0) = \frac{\Gamma(-2\nu)}{\Gamma(\alpha_4)\Gamma(\beta_4)}C_{3,0}(\rho; W) + \frac{\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}v_0(\rho; W), \\
&v_0(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}(1-x)^{1/4-\nu}\mathcal{F}(\alpha_4, \beta_4; \gamma_4; 1-x), \\
&C_{4,0}(\rho; W) = 2\lim_{\delta\to 0}\partial_\delta C_{1,0,\delta}^<(\rho; W), \\
&C_{1,0,\delta}(r; W) = \frac{R_C^2\sqrt{r}}{R_C^2 - r^2}x^{\delta/2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_{1\delta}, \beta_{1\delta}; \gamma_{1\delta}; x), \\
&C_{3,0}(\rho; W) = \frac{R_C^2\sqrt{\rho}}{R_C^2 - \rho^2}(1-x)^{1/4+\nu}\mathcal{F}(\alpha_1, \beta_1; \gamma_3; 1-x) = \\
&= B_0(W)C_{1,0}(\rho; W) - C_0(W)C_{4,0}(\rho; W),\ C_0(W) = \frac{\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)}, \\
&B_0(W) = C_0(W)f_0(W),\ f_0(W) = [2\psi(1) - \psi(\alpha_1) - \psi(\beta_1)]\,.
\end{aligned}$$

$$\begin{aligned}
&\alpha_{1\delta} = 1/2 + \delta/2 + \nu + \sigma,\ \beta_{1\delta} = 1/2 + \delta/2 + \nu - \sigma,\ \gamma_{1\delta} = 1 + \delta \\
&\alpha_1 = 1/2 + \nu + \sigma,\ \beta_1 = 1/2 + \nu - \sigma, \\
&\alpha_4 = 1/2 - \nu + \sigma,\ \beta_4 = 1/2 - \nu - \sigma.
\end{aligned}$$

Note that $C_{1,0}^<(\rho; W)$ and $C_{4,0}^<(\rho; W)$ are real-entire in W.

3.6.2 Asymptotics, $\rho \to 0$ ($x \to 0$)

We have

$$\begin{aligned}
&C_{1,0}(\rho; W) = C_{1,0\text{as}}(\rho)(1 + O(\rho)),\ C_{4,0}(\rho; W) = C_{4,0\text{as}}(\rho)\,(1 + O(\rho))\,, \\
&C_{1,0\text{as}}(\rho) = \rho^{1/2},\ C_{4,0\text{as}}(\rho) = \rho^{1/2}\ln\left(\frac{4\rho}{R_C}\right).
\end{aligned}$$

$$C_{3,0}(\rho;W) = .C_0(W)\left[f_0(W)C_{1,0\text{as}}(\rho) - C_{4,0\text{as}}(\rho)\right]\left(1+O(\rho^2)\right),$$
$$\operatorname{Im}W > 0 \text{ or } W = 0.$$

3.6.3 Asymptotics, $\Delta = R - \rho \to 0$ $(\delta = 1 - x \to 0)$

We have

$$C_{3,0}(\rho;W) = 2^{-3/2-2\nu}R_C^{1-2\nu}\Delta^{-1/2+2\nu}(1+O(\Delta)),$$

$$C_{1,0}(\rho;W) = \frac{\Gamma(2\nu)}{\Gamma(\alpha_1)\Gamma(\beta_1)}2^{-3/2+2\nu}R_C^{1+2\nu}\Delta^{-1/2-2\nu}(1+O(\Delta)),$$
$$\operatorname{Im}W > 0 \text{ or } W = 0.$$

3.6.4 Wronskian

We have

$$\operatorname{Wr}(C_{1,0}, C_{4,0}) = 1,$$
$$\operatorname{Wr}(C_{1,0}, C_{3,0}) = -\frac{\Gamma(\gamma_3)}{\Gamma(\alpha_1)\Gamma(\beta_1)} = -C_0(W).$$

Note that any solutions of eq. (3.23) are s.-integrable in the origin and only one solution $(C_{3,0})$ is s.-integrable on the right end R_C (for $\operatorname{Im}W > 0$), such that there is one solution $(C_{3,0})$ belonging to $L^2(0, R_C)$ for $\operatorname{Im}W > 0$ and the deficiency indexes of the symmetric operator $\check{h}_0$ (see below) are equal to $(1,1)$.

3.6.5 Symmetric operator $\hat{h}_0$

A symmetric operator $\hat{h}_0$ is defined by eq. (3.10) with $m = 0$.

3.6.6 Adjoint operator $\hat{h}_0^+ = \hat{h}_0^*$

It is easy to prove by standard way that the adjoint operator $\hat{h}_0^+$ coincides with the operator $\hat{h}_0^*$,

$$\hat{h}_0^+ : \begin{cases} D_{h_0^+} = D^*_{\check{h}_0}(0, R_C) = \{\psi_*, \psi_*' \text{ are a.c. in } (0, R_C),\ \psi_*, \hat{h}_0^+\psi_* \in L^2(0, R_C)\} \\ \hat{h}_0^+\psi_*(\rho) = \check{h}_0\psi_*(\rho),\ \forall\psi_* \in D_{h_0^+} \end{cases}.$$

3.6.7 Asymptotics

Because $\check{h}_0\psi_* \in L^2(0, R)$, we have

$$\check{h}_0\psi_*(\rho) = \eta(\rho),\ \eta \in L^2(0, R_C),$$

and we can represent ψ_* in the form

$$\psi_*(\rho) = c_1C_{1,0}(\rho;0) + c_2C_{3,0}(\rho;0) + I(\rho),$$
$$\psi_*'(\rho) = c_1\partial_\rho C_{1,0}(\rho;0) + c_2\partial_\rho C_{3,0}(\rho;0) + I'(\rho),$$

where

$$I(\rho)=\frac{C_{3,0}(\rho;0)}{C_0(0)}\int_0^\rho C_{1,0}(y;0)\eta(y)dy-\frac{C_{1,0}(\rho;0)}{C_0(0)}\int_0^\rho C_{3,0}(y;0)\eta(y)dy,$$
$$I'(\rho)=\frac{\partial_\rho C_{3,0}(\rho;0)}{C_0(0)}\int_0^\rho C_{1,0}(y;0)\eta(y)dy-\frac{\partial_\rho C_{1,0}(\rho;0)}{C_0(0)}\int_0^\rho C_{3,0}(y;0)\eta(y)dy.$$

I) $\rho\to 0$

We obtain with the help of the CB-inequality:

$$I(\rho)=O(\rho^{3/2}\ln\rho),\ I'(\rho)=O(\rho^{1/2}\ln\rho),$$

such that we have

$$\psi_*(\rho)=c_1C_{1,0\mathrm{as}}(\rho)+c_2C_{4,0\mathrm{as}}(\rho)+O(\rho^{3/2}\ln\rho),$$
$$\psi_*'(\rho)=c_1C_{1,0\mathrm{as}}'(\rho)+c_2C_{4,0\mathrm{as}}'(\rho)+O(\rho^{1/2}\ln\rho).$$

II) $\rho\to R_C$

In this case, we prove that $[\psi_*,\chi_*]^{R_C}=0$, $\forall\psi_*,\chi_*\in D_{h_m^+}$, such that we have

$$\Delta_{h_0^+}(\psi_*)=(\overline{c_1}c_2-\overline{c_2}c_1)=-\frac{i}{2}(.\overline{c_+}c_+-\overline{c_-}c_-),\ c_\pm=c_1\pm ic_2.$$

3.6.8 Self-adjoint hamiltonians $\hat{h}_{0\theta}$

The condition $\Delta_{h_0^+}(\psi)=0$ gives

$$c_-=-e^{2i\theta}c_+,\ |\theta|\le\pi/2,\ \theta=-\pi/2\sim\theta=\pi/2,\ \Longrightarrow$$
$$\Longrightarrow\ c_1\cos\theta=c_2\sin\theta,$$

or

$$\psi(\rho)=C\psi_{\theta\mathrm{as}}(\rho)+O(\rho^{3/2}\ln\rho),\ \psi'(\rho)=C\psi_{\theta\mathrm{as}}'(\rho)+O(\rho^{1/2}\ln\rho), \tag{3.24}$$
$$\psi_{\theta\mathrm{as}}(\rho)=C_{1,0\mathrm{as}}(\rho)\sin\theta+C_{4,0\mathrm{as}}(\rho)\cos\theta$$

We thus have a family of s.a. hamiltohians $\hat{h}_{0\theta}$,

$$\hat{h}_{0\theta}:\left\{\begin{array}{l} D_{h_{0\theta}}=\{\psi\in D_{h_0^+},\ \psi\text{ satisfy the boundary condition (3.24)}\\ \hat{h}_{0\theta}\psi=\check{h}_0\psi,\ \forall\psi\in D_{h_{0\theta}}\end{array}\right. .$$

3.6.9 The guiding functional

As a guiding functional $\Phi_{0\theta}(\xi;W)$ we choose

$$\Phi_{0\theta}(\xi;W)=\int_0^R U_{0\theta}(\rho;W)\xi(\rho)d\rho,\ \xi\in\mathbb{D}_\theta=D_r(0,R_C)\cap D_{h_{0\theta}}.$$
$$.U_{0\theta}(\rho;W)=C_{1,0}(\rho;W)\sin\theta+C_{4,0}(\rho;W)\cos\theta,$$

$U_{0\theta}(\rho;W)$ is real-entire solution of eq. (3.23) satisfying the boundary condition (3.24).

The guiding functional $\Phi_{0\theta}(\xi;W)$ is simple and the spectrum of $\hat{h}_{0\theta}$ is simple.

3.6.10 Green function $G_{0\theta}(\rho, y; W)$, spectral function $\sigma_{0\theta}(E)$

We find the Green function $G_{0\theta}(\rho, y; W)$ as the kernel of the integral representation

$$\psi(\rho) = \int_0^{R_C} G_{0\theta}(\rho, y; W)\eta(y)dy,\ \eta \in L^2(\mathbb{R}_+),$$

of unique solution of an equation

$$(\hat{h}_{0\theta} - W)\psi(\rho) = \eta(\rho),\ \mathrm{Im}\, W > 0, \tag{3.25}$$

for $\psi \in D_{h_{0\theta}}$. General solution of eq. (3.25) (under condition $\psi \in L^2(0, R)$) can be represented in the form

$$\psi(\rho) = aC_{3,0}(\rho; W) + \frac{C_{1,0}(\rho; W)}{C_0(W)}\eta_3(W) + I(\rho),\ \eta_3(W) = \int_0^{R_C} C_{3,0}(y; W)\eta(y)dy,$$
$$I(\rho) = \frac{C_{3,0}(\rho; W)}{C_0(W)}\int_0^\rho C_{1,0}(y; W)\eta(y)dy - \frac{C_{1,0}(\rho; W)}{C_0(W)}\int_0^\rho C_{3,0}(y; W)\eta(y)dy,$$
$$I(\rho) = O\left(\rho^{3/2}\ln\rho\right),\ \rho \to 0.$$

A condition $\psi \in D_{h_{0\theta}}$, (i.e.ψ satisfies the boundary condition (2.27)) gives

$$a = -\frac{\cos\theta}{C_0^2(W)\omega_\theta(W)}\eta_3(W),\ \omega_\theta(W) = f_0(W)\cos\zeta + \sin\zeta,$$

$$G_{0\theta}(\rho, y; W) = \pi\Omega_{0\theta}(W)U_{0\theta}(\rho; W)U_{0\theta}(y; W) +$$
$$+\begin{cases} \tilde{U}_{0\theta}(\rho; W)U_{0\theta}(y; W),\ \rho > y \\ U_{0\theta}(\rho; W)\tilde{U}_{0\theta}(y; W),\ \rho < y \end{cases}, \tag{3.26}$$
$$\Omega_{0\theta}(W) = \frac{\tilde{\omega}_\theta(W)}{\pi\omega_\theta(W)},\ \tilde{\omega}_\theta(W) = f_0(W)\sin\theta - \cos\zeta,$$
$$\tilde{U}_{0\theta}(\rho; W) = C_{1,0}^<(\rho; W)\cos\theta - C_{4,0}^<(\rho; W)\sin\theta,$$

where we used an equality

$$C_{3,0}^<(\rho; W) = C_0(W)[\tilde{\omega}_\theta(W)U_{0\theta}(\rho; W) + \omega_\theta(W)\tilde{U}_{0\theta}(\rho; W)].$$

Note that the functions $U_{0\theta}(\rho; W)$ and $\tilde{U}_{0\theta}(\rho; W)$ are real-entire in W and the last term in the r.h.s. of eq. (3.26) is real for $W = E$. For $\sigma_{0\theta}'(E)$, we find

$$\sigma_{0\theta}'(E) = \mathrm{Im}\,\Omega_{0\theta}(E + i0).$$

3.6.11 Spectrum

3.6.12 $W = (1 + 4R_C g)/R_C^2 + \tilde{\Delta}$, $\tilde{\Delta} \sim 0$

A direct estimation gives

$$\Omega_{0\zeta}(W) = \begin{cases} \Omega_{0\zeta}(W_0) + O(\sqrt{\tilde{\Delta}}),\ g = g_{0k} \text{ or } g \neq g_{0k},\ \zeta \neq \zeta_0 \\ O(1/\sqrt{\tilde{\Delta}}),\ g \neq g_{0k},\ \zeta = \zeta_0 \end{cases},$$

$R_C g_{0,k} = -N_{0,k}^2$, $N_{0,k} = 1 + 2k, \tan\theta_0 = -f_{0|0} = f_0(W)|_{\tilde{\Delta}=0}$, $\mathrm{Im}[\Omega_{0\zeta}(W_0)] = 0$. This result means that the levels with $E = E_0$ are absent.

3.6.13 $\theta = \pi/2$

First we consider the case $\theta = \pi/2$.

In this case, we have $U_{0\pi/2}(\rho;W) = C_{1,0}(\rho;W)$ and

$$\sigma'_{0\pi/2}(E) = \frac{1}{\pi}\operatorname{Im} f_0(E+i0).$$

All results for spectrum and spectral function can be obtained from the corresponding results of subsec. 4.7 by setting there $m = 0$.

$w = R_C^2 E > 1 + 4R_C g$ In this case, $f_0(E)$ is a finite complex function and we find

$$\sigma'_{0\pi/2}(E) = \frac{1}{\pi}\operatorname{Im}\mathcal{V}_0(E) \equiv \varrho^2_{0\pi/2}(E) > 0.$$

where $f_0(E) = \mathcal{U}_0(E) + i\mathcal{V}_0(E)$, $\mathcal{U}_0(E) = \operatorname{Re} f_0(E)$, $\mathcal{V}_0(E) = \operatorname{Im} f_0(E) > 0$. The spectrum of $\hat{h}_{0\pi/2}$ is simple and continuous, $\operatorname{spec}\hat{h}_{0\pi/2} = [(1+4R_C g)/R_C^2, \infty)$, and

$$\sigma'_{0\pi/2}(E) = \left\{\begin{array}{c} O(\sqrt{\Delta}),\ g \neq g_{0,k} \\ O(1/\sqrt{\Delta}),\ g = g_{0,k} \end{array}\right.,\ \Delta \to +0,$$

$\Delta = E - (1+4R_C g)/R_C^2$.

$w = R_C^2 E \leq 1 + 4R_C g$**,** $w < 1$ In this case, we have

$$\sigma'_{0\pi/2}(E) = \sum_{n=0}^{n_{\max}} Q^2_{0\pi/2,n}\delta(E - \mathcal{E}_{0,n}),\ Q_{0\pi/2,n} = \frac{2}{R_C}\sqrt{\frac{R_C^2 g^2 - N_{0,n}^4}{N_{0,n}^3}},$$

$$\mathcal{E}_{0,n} = .\frac{1 - (R_C|g|/N_{0,n} + N_{0,n})^2}{R_C^2},\ R_C g < R_C g_{0,0} = -N_{1,0}^2 = -1.$$

The spectrum of $\hat{h}_{0\pi/2}$ is discrete and simple and has the form

$$\operatorname{spec}\hat{h}_{0\pi/2} = \{\mathcal{E}_{0,n},\ \mathcal{E}_{0,n} < 1 - 4R_C|g|,\ n = 0,1,...,n_{\max}\},$$

$n_{\max} = k$ for $\sqrt{R_C|g|} = 1 + 2(k+\delta)$, $0 < \delta \leq 1$. The discrete part of the spectrum is absent for $g \geq g_{0,0} = -N_{0,0}^2/R_C = -1/R_C$.

$w = R_C^2 E \leq 1 + 4R_C g$**,** $w \geq 1$ In this case, we have $\sigma'_{0\pi/2}(E) = 0$.

Finally, we find.

The spectrum of $\hat{h}_{0\pi/2}$ is simple, $\operatorname{spec}\hat{h}_{0\pi/2} = [1+4R_C g)/R_C^2, \infty) \cup \{\mathcal{E}_{0,n},\ n = 0,1,...n_{\max}\}$, the discrete part of spectrum is present for $g < g_{0,0}$. The set of functions

$$\big\{U_{0\pi/2,E}(\rho) = \varrho_{0,\pi/2}(E)C_{1,0}(\rho;E),\ E \geq (1+4R_C g)/R_C^2; U_{0\pi/2,n}(\rho) = Q_{0\pi/2,n}C_{1,0}(\rho;\mathcal{E}_{0,n}),$$

$$n = 0,1,...n_{\max}\}$$

forms a complete orthogonalized system in $L^2(0,R_C)$.

The same results we obtain for the case $\zeta = -\pi/2$.

3.6.14 $|\theta| < \pi/2$

Now we consider the case $|\theta| < \pi/2$.

In this case, we can represent $\sigma'_\theta(E)$ in the form

$$\sigma'_{0\theta}(E) = -\frac{1}{\pi\cos^2\theta}\operatorname{Im}\frac{1}{f_{0\theta}(E+i0)},\ f_{0\theta}(W) = f_0(W) + \tan\zeta.$$

$E > (1+4R_C g)/R_C^2/$ In this case, we have

$$\sigma'_{0\theta}(E) = \frac{1}{\pi}\frac{\mathcal{V}_0(E)}{[\mathcal{U}_0(E)\cos\theta + \sin\theta]^2 + \mathcal{V}_0^2(E)\cos^2\theta} \equiv \rho^2_{0\theta}(E).$$

The spectrum of $\hat{h}_{0\theta}$ is simple and continuous, $\operatorname{spec}\hat{h}_{0\theta} = [(1+4R_C g)/R_C^2, \infty)$.

$w = 1 + 4R_C g + \Delta$**,** $\Delta \sim 0$ In this case, we have

$$\alpha_1 = 1/2 + \left\{\begin{array}{l} -i\sqrt{R_C g}/2 + \nu + O(\Delta),\ g > 0 \\ 2\nu,\ g = 0 \\ \sqrt{R_C|g|}/2 + \nu + O(\Delta),,\ g < 0 \end{array}\right.,$$

$$\beta_1 = 1/2 + \left\{\begin{array}{l} i\sqrt{R_C g}/2 + \nu + O(\Delta),\ g > 0 \\ 0,\ g = 0 \\ -\sqrt{R_C|g|}/2 + \nu + O(\Delta),,\ g < 0 \end{array}\right..$$

A direct estimation gives

$$\sigma'_{0\theta}(E) = \left\{\begin{array}{l} \left\{\begin{array}{l} O(\sqrt{\Delta}),\ g = g_{0k} \text{ or } g \neq g_{0k},\ \theta \neq \theta_0 \\ O(1/\sqrt{\Delta}),\ g \neq g_{0k},\ \theta = \theta_0 \end{array}\right.,\ \Delta > 0 \\ 0,\ \Delta < 0 \end{array}\right.,$$

where $\tan\theta_0 = -f_{0|0} = -\left.f_0(E)\right|_{\Delta=0}$. Note that the discrete level with $E = (1+4R_C g)R_C^2$ is absent.

$E < (1+4R_C g)/R_C^2$ In this case, we have $\operatorname{Im}\nu|_{W=E} = 0,\ \nu|_{W=E} > 0$,

$$\alpha_1 = 1/2 + \nu - i\sqrt{w-1}/4,\ \beta_1 = \overline{\alpha_1},\ w \geq 1,$$
$$\alpha_1 = 1/2 + \nu + \sqrt{1-w}/4,\ \beta_1 = 1/2 + \nu - \sqrt{1-w}/4,\ w < 1.$$

Thus, we have: the function $[f_{0\theta}(E)]^{-1}$ is real except the points $E_{0n}(\theta)$,

$$f_{0\theta}(E_{0n}(\theta)) = 0, \tag{3.27}$$

such that we obtain

$$\sigma'_{0\theta}(E) = \sum_{n\in\mathcal{N}_0} Q^2_{0\theta,n}\delta(E - E_{0n}(\theta)),\ Q_{0\theta,n} = \frac{1}{\sqrt{\partial_E f_{0\theta}(E_{on}(\theta))}\cos\theta},$$
$$\partial_E f_0(E) > 0,$$

where $\mathcal{N}_0$ is a subset of $\mathbb{Z}$ to be described below. Furthermore, we find

$$\partial_\theta E_{0n}(\theta) = -[\partial_E f_0(E_{0n}(\theta))\cos^2\theta]^{-1} < 0.$$

$g \geq g_{0,0} = -1/R_C$ In this case, the function $f_0(E)$ has the properties: $f_0(E) \to -\infty$ as $E \to -\infty$; $f_0(E) \to f_{0|0} - 0 = -\tan\theta_0 - 0$ as $E \to (1+4R_C g)R_C^2 - 0$;.$f_0(E)$ increases monotonocally on the interval $(-\infty, (1+4R_C g)R_C^2)$. Then we find: in the interval $(-\infty, (1+4R_C g)R_C^2)$, for any fixed $\theta \in (\theta_0.\pi/2)$, there is one level $E_{0,-1}(\theta)$ which run monotonocally from $-\infty$ to $(1+4R_C g)R_C^2 - 0$ as θ run from $\pi/2 - 0$ to $\theta_0 + 0$; there are no discrete levels on the interval $(-\infty, (1+4R_C g)R_C^2)$ for $\theta \in (-\pi/2, \theta_0)$. Formally, eq. (3.27) has solution $E_{0,-1}(\theta_0) = (1+4R_C g)R_C^2$ for $\theta = \theta_0$. However, as was noted above, such levels are absent We find also

$$\mathcal{N}_0 = \mathcal{N}_{0,-1} = \left\{ \begin{array}{l} \{-1\},\ \theta \in (\theta_0, \pi/2) \\ \varnothing,\ \theta \in (-\pi/2, \theta_0] \end{array} \right. .$$

$g_{0,k+1} \leq g < g_{0,k+1}$**,** $\sqrt{R_C|q|} = 1/2 + k + \delta$**,** $0 < \delta \leq 1$**,** $k \in \mathbb{Z}_+$ In this case, the function $f_0(E)$ has the properties: $f_0(E) \to -\infty$ as $E \to -\infty$; $f(\mathcal{E}_{0n} \pm 0) = \mp\infty$ $n = 0, 1, ..., n_{\max} = k$; $f_0(E) \to f_{0|0}(E) - 0 = -\tan\theta_0 - 0$ as $E \to (1+4R_C g)/R_C^2 - 0$. We find: in each interval $(\mathcal{E}_{0n-1}, \mathcal{E}_{0n})$, $n = 0, ..., k$ (we set $\mathcal{E}_{0,-1} = -\infty$), for any fixed $\theta \in (-\pi/2, \pi/2)$, there is one level $E_{0n}(\theta)$ which run monotonocally from $\mathcal{E}_{n-1} + 0$ to $\mathcal{E}_n - 0$ as ζ run from $\pi/2 - 0$ to -$\pi/2 + 0$; in the interval $(\mathcal{E}_k, (1+4R_C g)R_C^2)$, for any fixed $\theta \in (\theta_0, \pi/2)$, there is one level $E_{0k+1}(\theta)$ which run monotonocally from $\mathcal{E}_k + 0$ to $(1+4R_C g)R_C^2 - 0$ as ζ run from $\pi/2 - 0$ to $\theta_0 + 0$; thare are no levels in the interval $(\mathcal{E}_k, (1+4R_C g)/R_C^2)$ for $\theta \in (-\pi/2, \theta_0)$. Formally, eq. (3.22) has solution $E_{0,k+1}(\theta_0) = (1+4R_C g)/R_C^2$ for $\theta = \theta_0$. However, as was noted above, such levels are absent.. We find also

$$\mathcal{N}_0 = \mathcal{N}_{0,k}(\theta) = \left\{ \begin{array}{l} \{0, 1, ..., k\},\ \theta \in (-\pi/2, \theta_0] \\ \{0, 1, ..., k+1\},\ \zeta \in (\theta_0, \pi/2) \end{array} \right. .$$

Finally, we obtain. The spectrum of $\hat{h}_{0\theta}$ is simple and spec$\hat{h}_{0\theta} = [(1+4R_C g)/R_C^2, \infty) \cup \{E_{0n}(\theta),\ n \in \mathcal{N}_0\}$. The set of functions

$$\left\{U_{0\theta,E}(\rho) = \varrho_{0\theta}(E)U_{0\theta}(\rho; E),\ E \geq (1+4R_C g)/R_C^2;\ U_{0\theta,n}(\rho) = Q_{0\theta,n}U_{0\theta}(\rho; E_{0\theta}(\zeta),),\ n \in \mathcal{N}_0\right\}$$

forms a complete orthogonalized system in $L^2(0, R_C)$.

4 Conclusions

As we found, two dimensional oscillator and coulomb problems on pseudoshpere are described by the same equations in terms of the variables α and β This means that each point of the spectra of one of these theories corresponds a point of the spectra of the other theory, i.e. there is one-to-one correspondence between points of the the planes E_O, λ and E_C, g.

5 Acknowledgement

I.Tyutin thanks RFBR Grant 11-01-00830 for partial support.

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